INFINITESIMAL LYAPUNOV FUNCTIONS FOR SINGULAR FLOWS

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Abstract. We present an extension of the notion of infinitesimal Lyapunov function to singular flows, and from this technique we deduce a characterization of partial/sectional hyperbolic sets. In absence of singularities, we can also characterize uniform hyperbolicity.

These conditions might be expressed using the vector field X and its space derivative DX together with an infinitesimal Lyapunov function only and are reduced to checking that a certain symmetric operator is positive definite on the trapping region.

1. Introduction

The hyperbolic theory of dynamical systems is now almost a classical subject in mathematics and one of the main paradigms in dynamics. Developed in the 1960s and 1970s after the work of Smale, Sinai, Ruelle, Bowen [37, 36, 10, 11], among many others, this theory deals with compact invariant sets Λ for diffeomorphisms and flows of closed finite-dimensional manifolds having a hyperbolic splitting of the tangent space. That is, $T_\Lambda M = E^s \oplus E^X \oplus E^u$ is a continuous splitting of the tangent bundle over Λ, where E^X is the flow direction, the subbundles are invariant under the derivative DX_t of the flow X_t

$$DX_t \cdot E^*_x = E^*_{X_t(x)}, \quad x \in \Lambda, \quad t \in \mathbb{R}, \quad * = s, X, u;$$

E^s is uniformly contracted by DX_t and E^u is likewise expanded: there are $K, \lambda > 0$ so that

$$\|DX_t \mid E^s_x\| \le Ke^{-\lambda t}, \quad \|(DX_t \mid E^u_x)^{-1}\| \le Ke^{-\lambda t}, \quad x \in \Lambda, \quad t \in \mathbb{R}.$$

Very strong properties can be deduced from the existence of such hyperbolic structure; see for instance [10, 11, 35, 17, 33].

More recently, extensions of this theory based on weaker notions of hyperbolicity, like the notions of dominated splitting, partial hyperbolicity, volume hyperbolicity and singular hyperbolicity (for three-dimensional flows) have been developed to encompass larger classes of systems beyond the uniformly hyperbolic ones; see [6] and specifically [41, 4] for singular hyperbolicity and Lorenz-like attractors.

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One of the technical difficulties in this theory is to actually prove the existence of a hyperbolic structure, even in its weaker forms. We mention that Malkus showed that the Lorenz equations, presented in [20], are the equations of motion of a waterwheel, which was built at MIT in the 1970s and helped to convince the skeptical physicists of the reality of chaos; see [38, Section 9.1]. Only around the year 2000 was it established by Tucker in [39, 40] that the Lorenz system of equations, with the parameters indicated by Lorenz, does indeed have a chaotic strange attractor. This proof is a computer assisted proof which works for a specific choice of parameters, and has not been improved to this day. More recently, Hunt and Mackay in [15] have shown that the behavior of a certain physical system, for a specific choice of parameters which can be fixed in a concrete laboratory setup, is modeled by an Anosov flow.

The most usual and geometric way to prove hyperbolicity is to use a field of cones. This idea goes as far back as the beginning of the hyperbolic theory; see Alekseev [1, 2, 3]. Given a continuous splitting $T_\Lambda M = E \oplus F$ of the tangent space over an invariant subset Λ, not necessarily invariant with respect to a flow X_t, a field of cones of size $a > 0$ centered around F is the family

$$C_a(x) := \{\vec{0}\} \cup \{(u, v) \in E_x \times F_x : \|u\| \leq a\|v\|\}, \quad x \in \Lambda.$$

Let us assume that there exists $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ and every negative t

(1) $\overline{DX_t \cdot C_a(x)} \subset C_a(X_t(x))$ (the overline denotes closure in $T_{X_t(x)}M$);
(2) $\|DX_t \cdot w\| \leq \lambda\|w\|$ for each $v \in C_a(x)$.

Then there exists an invariant bundle E^s contained in the cone field C_a over Λ whose vectors are uniformly contracted. The complementary cone field satisfies the analogous to the first item above for positive t. This ensures the existence of a partially hyperbolic splitting over Λ.

We present a simple extension of the notion of infinitesimal Lyapunov function, from [16], to singular flows, and show how this technique provides a new characterization of partially hyperbolic structures for invariant sets for flows, and also of singular and sectional hyperbolicity. In the absence of singularities, we can also rephrase uniform hyperbolicity with the language of infinitesimal Lyapunov functions.

This technique is not new. Lewowicz used it in his study of expansive homeomorphisms [19] and Wojtkowski adapted it for the study of Lyapunov exponents in [43]. Using infinitesimal Lyapunov functions, Wojtkowski was able to show that the second item above is superfluous: the geometric condition expressed in the first item is actually enough to conclude uniform contraction.

Workers using these techniques, like Lewowicz [18], Markarian [23], Wojtkowski [45], Burns-Katok [16], have only considered either dynamical systems given by maps or by flows without singularities. In this last case, the authors deal with the Linear Poincaré Flow on the normal bundle to the flow direction.

We adapt ideas introduced first by Lewowicz in [18], and developed by several other authors in different contexts, to the setting of vector bundle automorphisms over flows with singularities; see also [46, 44, 23] for other known applications of this technique to

billiards and symplectic flows, and also [31, 30] for a general theory of $\mathcal{J}$-monotonous linear transformations.

We improve on these results by showing, roughly, that the condition on item (1) above on a trapping region for a flow implies partial hyperbolicity, even when singularities are present. This also provides a way to define uniform hyperbolicity on a compact invariant set for a smooth flow generated by a vector field X, using only X and DX together with a family of Lyapunov functions.

We then provide an extra necessary and sufficient condition ensuring that the complementary cone, containing invariant subbundle E^c, which contains the flow direction, is such that the area form along any two-dimensional subspace of E^c is uniformly expanded by the action of the tangent cocycle DX_t of the flow X_t (this property is today known as sectional-hyperbolicity; see [24]).

Moreover, these conditions can be expressed using the vector field X and its space derivative DX together with an infinitesimal Lyapunov function, and are reduced to checking that a certain symmetric operator is positive definite on all points of the trapping region. While we usually define hyperbolicity by using the differential Df of a diffeomorphism f, or the cocycle $(DX_t)_{t\in\mathbb{R}}$ associated to the continuous one-parameter group $(X_t)_{t\in\mathbb{R}}$, we show how to express partial hyperbolicity using only the interplay between the infinitesimal generator X of the group X_t, its derivative DX and the infinitesimal Lyapunov function. Since in many situations dealing with mathematical models from the physical, engineering or social sciences, it is the vector field that is given and not the flow, we expect that the theory here presented to be useful to develop simpler algorithms to check hyperbolicity.

1.1. Preliminary definitions.

Before the main statements we collect some definitions in order to state the main results.

Let M be a connected compact finite n-dimensional manifold, $n \geq 3$, with or without boundary, together with a flow $X_t : M \to M, t \in \mathbb{R}$ generated by a C^1 vector field $X : M \to TM$, such that X is inwardly transverse to the boundary ∂M, if $\partial M \neq \emptyset$.

An *invariant set* Λ for the flow of X is a subset of M which satisfies $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. The *maximal invariant set of the flow* is $M(X) := \cap_{t\geq 0} X_t(M)$, which is clearly a compact invariant set.

A *trapping region* U for a flow X_t is an open subset of the manifold M which satisfies: $X_t(U)$ is contained in U for all $t > 0$; and there exists $T > 0$ such that $\overline{X_t(U)}$ is contained in the interior of U for all $t > T$.

A *singularity* for the vector field X is a point $\sigma \in M$ such that $X(\sigma) = \vec{0}$ or, equivalently, $X_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. The set formed by singularities is the *singular set of* X denoted $\mathrm{Sing}(X)$. We say that a *singularity is hyperbolic* if the eigenvalues of the derivative $DX(\sigma)$ of the vector field at the singularity σ have nonzero real part.

Definition 1. A *dominated splitting* over a compact invariant set Λ of X is a continuous DX_t-invariant splitting $T_\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ and such that there are positive constants K, λ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \text{ for all } x \in \Lambda, \text{ and all } t > 0. \tag{1.1}$$

A compact invariant set Λ is said to be *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E \oplus F$ such that subbundle E is uniformly contracted. In this case F is the *central subbundle* of Λ.

A compact invariant set Λ is said to be *singular-hyperbolic* if it is partially hyperbolic and the action of the tangent cocycle expands volume along the central subbundle, i.e.,

$$|\det(DX_t|_{F_x})| > Ce^{\lambda t}, \forall t > 0,\ \forall\ x \in \Lambda. \tag{1.2}$$

Definition 2. We say that a DX_t-invariant subbundle $F \subset T_\Lambda M$ is a *sectionally expanding* subbundle if $\dim F_x \geq 2$ is constant for $x \in \Lambda$ and there are positive constants C, λ such that for every $x \in \Lambda$ and every two-dimensional linear subspace $L_x \subset F_x$ one has

$$|\det(DX_t|_{L_x})| > Ce^{\lambda t}, \forall t > 0. \tag{1.3}$$

And, finally, the definition of sectional-hyperbolicity.

Definition 3. [24, Definition 2.7] A *sectional-hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and the central subbundle is sectionally expanding.

We note that a sectional-hyperbolic set always is singular-hyperbolic, however the reverse is only true in dimension three, not in higher dimensions; see for instance [24].

Remark 1.1. The properties of sectional-hyperbolicity can be expressed in the following equivalent forms; see [4]. There exists $T > 0$ such that

- $\|DX^T|_{E_x}\| < \frac{1}{2}$ for all $x \in \Lambda$ (uniform contraction); and
- $|\det(DX^T|_{F_x})| > 2$ for all $x \in \Lambda$ and each 2-subspace F_x of E^c_x (2-sectional expansion).

We say that a compact invariant set Λ is a *volume-hyperbolic* set if it has a dominated splitting $E \oplus F$ such that the volume along its subbundles is uniformly contracted (along E) and expanded (along F) by the action of the tangent cocyle. If the whole manifold M is a volume-hyperbolic set for a flow X_t, then we say that X_t is a volume-hyperbolic flow.

We recall that a flow X_t is said to be *Anosov* if the whole manifold M is a hyperbolic set for the flow. Based on this definition, we say that X_t is a *sectional-Anosov flow* if the maximal invariant set $M(X)$ is a sectional-hyperbolic set for the flow.

From now on, we consider M a connected compact finite dimensional riemannian manifold, $U \subset M$ a trapping region, $\Lambda(U) = \Lambda_X(U) := \cap_{t>0}\overline{X_t(U)}$ the maximal positive invariant subset in U for the vector field X and E_U a finite dimensional vector bundle over U. We also consider that all singularities of X in U (if they exist) are hyperbolic.

1.1.1. *Fields of quadratic forms; positive and negative cones.* Let E_U be a finite dimensional vector bundle with base U and $\mathcal{J} : E_U \to \mathbb{R}$ be a continuous family of quadratic forms $\mathcal{J}_x : E_x \to \mathbb{R}$ which are non-degenerate and have index $0 < q < \dim(E) = n$. The index q of $\mathcal{J}$ means that the maximal dimension of subspaces of non-positive vectors is q.

We also assume that $(\mathcal{J}_x)_{x\in U}$ is continuously differentiable along the flow. The continuity assumption on $\mathcal{J}$ means that for every continuous section Z of E_U the map $U \ni x \mapsto \mathcal{J}(Z(x)) \in \mathbb{R}$ is continuous. The C^1 assumption on $\mathcal{J}$ along the flow means that the map

$\mathbb{R} \ni t \mapsto \mathcal{J}_{X_t(x)}(Z(X_t(x))) \in \mathbb{R}$ is continuously differentiable for all $x \in U$ and each C^1 section Z of E_U.

We let $\mathcal{C}_\pm = \{C_\pm(x)\}_{x\in U}$ be the family of positive and negative cones associated to $\mathcal{J}$

$$C_\pm(x) := \{0\} \cup \{v \in E_x : \pm\mathcal{J}_x(v) > 0\} \quad x \in U$$

and also let $\mathcal{C}_0 = \{C_0(x)\}_{x\in U}$ be the correspoing family of zero vectors $C_0(x) = \mathcal{J}_x^{-1}(\{0\})$ for all $x \in U$.

1.1.2. *Linear multiplicative cocycles over flows.* Let $A : E \times \mathbb{R} \to E$ be a smooth map given by a collection of linear bijections

$$A_t(x) : E_x \to E_{X_t(x)}, \quad x \in M, t \in \mathbb{R},$$

where M is the base space of the finite dimensional vector bundle E, satisfying the cocycle property

$$A_0(x) = Id, \quad A_{t+s}(x) = A_t(X_s(x)) \circ A_s(x), \quad x \in M, t, s \in \mathbb{R},$$

with $\{X_t\}_{t\in\mathbb{R}}$ a smooth flow over M. We note that for each fixed $t > 0$ the map $A_t : E \to E, v_x \in E_x \mapsto A_t(x) \cdot v_x \in E_{X_t(x)}$ is an automorphism of the vector bundle E.

The natural example of a linear multiplicative cocycle over a smooth flow X_t on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle TM of a finite dimensional compact manifold M.

The following definitions are fundamental to state our results.

Definition 4. Given a continuous field of non-degenerate quadratic forms $\mathcal{J}$ with constant index on the trapping region U for the flow X_t, we say that the cocycle $A_t(x)$ over X is

- $\mathcal{J}$-*separated* if $A_t(x)(C_+(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- *strictly* $\mathcal{J}$-*separated* if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$;
- $\mathcal{J}$-*monotone* if $\mathcal{J}_{X_t(x)}(A_t(x)v) \geq \mathcal{J}_x(v)$, for each $v \in T_xM \setminus \{\vec{0}\}$ and $t > 0$;
- *strictly* $\mathcal{J}$-*monotone* if $\partial_t\big(\mathcal{J}_{X_t(x)}(A_t(x)v)\big) \mid_{t=0} > 0$, for all $v \in T_xM \setminus \{0\}$, $t > 0$ and $x \in U$;
- $\mathcal{J}$-*isometry* if $\mathcal{J}_{X_t(x)}(A_t(x)v) = \mathcal{J}_x(v)$, for each $v \in T_xM$ and $x \in U$.

Thus, $\mathcal{J}$-separation corresponds to simple cone invariance and strict $\mathcal{J}$-separation corresponds to strict cone invariance under the action of $A_t(x)$.

We say that the flow X_t is (strictly) $\mathcal{J}$-*separated* on U if $DX_t(x)$ is (strictly) $\mathcal{J}$-*separated* on T_UM. Analogously, the flow of X on U is (strictly) $\mathcal{J}$-*monotone* if $DX_t(x)$ is (strictly) $\mathcal{J}$-*monotone*.

Remark 1.2. If a flow is strictly $\mathcal{J}$-separated, then for $v \in T_xM$ such that $\mathcal{J}_x(v) \leq 0$ we have $\mathcal{J}_{X^{-t}(x)}(DX_{-t}(v)) < 0$ for all $t > 0$ and x such that $X_{-s}(x) \in U$ for every $s \in [-t, 0]$. Indeed, otherwise $\mathcal{J}_{X^{-t}(x)}(DX^{-t}(v)) \geq 0$ would imply $\mathcal{J}_x(v) = \mathcal{J}_x\big(DX_t(DX^{-t}(v))\big) > 0$, contradicting the assumption that v was a non-positive vector.

This means that a flow X_t is strictly $\mathcal{J}$-separated if, and only if, its time reversal X_{-t} is strictly $(-\mathcal{J})$-separated.

A vector field X is $\mathcal{J}$-*non-negative* on U if $\mathcal{J}(X(x)) \geq 0$ for all $x \in U$, and $\mathcal{J}$-*non-positive* on U if $\mathcal{J}(X(x)) \leq 0$ for all $x \in U$. When the quadratic form used in the context is clear, we will simply say that X is non-negative or non-positive.

We say that a family $\mathcal{J}_0$ of quadratic forms on U is *equivalent* to $\mathcal{J}$, and we write $\mathcal{J} \sim \mathcal{J}_0$, if there exists $C > 1$ satisfying

$$\frac{1}{C} \cdot \mathcal{J}_0(v) \leq \mathcal{J}(v) \leq C \cdot \mathcal{J}_0(v), \quad v \in T_x M, v \neq \vec{0}, x \in U.$$

We note that equivalent quadratic forms have the same cones of positive, negative and zero vectors.

1.2. **Statement of the results.**

Theorem A. *A maximal invariant subset Λ of a trapping region U is a partially hyperbolic set for a flow X_t if, and only if, there is a C^1 field $\mathcal{J}$ of non-degenerate quadratic forms with constant index, equal to the dimension of the stable subspace of Λ, such that X_t is a non-negative strictly $\mathcal{J}$-separated flow on U.*

This is a direct consequence of a corresponding result for linear multiplicative cocycles over vector bundles which we state in Theorem 2.13 and prove in Section 2.4.

We obtain a criterion for partial and uniform hyperbolicity which extends the one given by Lewowicz, in [18], and Wojtkowski, in [46]. The condition of strict $\mathcal{J}$-separation can be expressed only using the vector field X and its spatial derivative DX, as follows.

Proposition 1.3. *A $\mathcal{J}$-non-negative vector field X on U is (strictly) $\mathcal{J}$-separated if, and only if, there exists an equivalent family $\mathcal{J}_0$ of forms and there exists a function $\delta : U \to \mathbb{R}$ such that the operator $\tilde{J}_{0,x} := J_0 \cdot DX(x) + DX(x)^* \cdot J_0$ satisfies*

$$\tilde{J}_{0,x} - \delta(x) J_0 \quad \text{is positive (definite) semidefinite}, \quad x \in U,$$

where $DX(x)^*$ is the adjoint of $DX(x)$ with respect to the adapted inner product.

Again this is a consequence of a corresponding result for linear multiplicative cocycles where DX is replaced by the infinitesimal generator

$$D(x) := \lim_{t \to 0} \frac{A_t(x) - Id}{t}$$

of the cocycle $A_t(x)$.

As a consequence of Theorem A we characterize hyperbolic maximal invariant subsets in trapping regions without singularities as follows. We recall that the index of a (partially) hyperbolic set is the dimension of the uniformly contracted subbundle of its tangent bundle.

Corollary B. *The maximal invariant subset Λ of U is a hyperbolic set for X of index s if, and only if, there exist $\mathcal{J}, \mathcal{G}$ smooth families of non-degenerate quadratic forms on U with constant index s and $n - s - 1$, respectively, where $s < n - 2$ and $n = dim(M)$, such that X_t is strictly $\mathcal{J}$-separated non-negative on U with respect to $\mathcal{J}$, X_t is strictly $\mathcal{G}$-separated non-positive with respect to $\mathcal{G}$, and there are no singularities of X in U.*

1.2.1. *Incompressible vector fields.* To state the next result, we recall that a vector field is said to be *incompressible* if its flow has null divergence, i.e., it is volume-preserving on M.

In this particular case, we have the following easy corollary of Theorem A, since a partially hyperbolic flow in a compact manifold must expand volume along the central direction. Moreover, if the stable direction has codimension 2, the central direction expands area.

Corollary C. *Let X be a C^1 incompressible vector field on a compact finite dimensional manifold M which is non-negative and strictly $\mathcal{J}$-separated for a family $\mathcal{J}$ of non-degenerate and indefinite quadratic forms with index* $\operatorname{ind}(\mathcal{J}) = \dim(M) - 2$. *Then X_t is an Anosov flow.*

Indeed, the results of Doering [12] and Morales-Pacifico-Pujals [26], in dimension three, Vivier [42] and Bautista-Morales [5], in higher dimensions, ensure that there are no singularities in the interior of a sectional-hyperbolic set, and so this set is hyperbolic; see Section 3.2.

To present the results about sectional-hyperbolicity, we need some more definitions.

1.2.2. *$\mathcal{J}$-monotonous Linear Poincaré Flow.* We apply these notions to the linear Poincaré flow defined on regular orbits of X_t as follows.

We assume that the vector field X is non-negative on U. Then, the span E^X_x of $X(x) \neq \vec{0}$ is a $\mathcal{J}$-non-degenerate subspace. According to item (1) of Proposition 2.1, this means that $T_xM = E^X_x \oplus N_x$, where N_x is the pseudo-orthogonal complement of E^X_x with respect to the bilinear form $\mathcal{J}$, and N_x is also non-degenerate. Moreover, by the definition, the index of $\mathcal{J}$ restricted to N_x is the same as the index of $\mathcal{J}$. Thus, we can define on N_x the cones of positive and negative vectors, respectively, N^+_x and N^-_x, just like before.

Now we define the Linear Poincaré Flow P^t of X_t along the orbit of x, by projecting DX_t orthogonally (with respect to $\mathcal{J}$) over $N_{X_t(x)}$ for each $t \in \mathbb{R}$:

$$P^t v := \Pi_{X_t(x)} DX_t v, \quad v \in T_xM, t \in \mathbb{R}, X(x) \neq \vec{0},$$

where $\Pi_{X_t(x)} : T_{X_t(x)}M \to N_{X_t(x)}$ is the projection on $N_{X_t(x)}$ parallel to $X(X_t(x))$. We remark that the definition of Π_x depends on $X(x)$ and $\mathcal{J}_X$ only. The linear Poincaré flow P^t is a linear multiplicative cocycle over X_t on the set U with the exclusion of the singularities of X.

In this setting we can say that the linear Poincaré flow is (strictly) $\mathcal{J}$-separated and (strictly) $\mathcal{J}$-monotonous using the non-degenerate bilinear form $\mathcal{J}$ restricted to N_x for a regular $x \in U$. More precisely: P^t is $\mathcal{J}$-monotonous if $\partial_t \mathcal{J}(P^t v) \mid_{t=0} \geq 0$, for each $x \in U, v \in T_xM \setminus \{\vec{0}\}$ and $t > 0$, and strictly $\mathcal{J}$-monotonous if $\partial_t \mathcal{J}(P^t v) \mid_{t=0} > 0$, for all $v \in T_xM \setminus \{\vec{0}\}$, $t > 0$ and $x \in U$.

Theorem D. *The maximal invariant subset Λ of the trapping region U is a sectional-hyperbolic set for X_t if, and only if, there is a C^1 field $\mathcal{J}$ of non-degenerate quadratic forms with constant index, equal to the dimension of the stable subspace of Λ, such that: X_t is a non-negative strictly $\mathcal{J}$-separated flow on U, whose singularities are sectionally hyperbolic with index* $\operatorname{ind}(\sigma) \geq \operatorname{ind}(\Lambda)$, *and for each compact invariant subset Γ of Λ*

without singularities there exists a family $\mathcal{J}_0$ of quadratic forms equivalent to $\mathcal{J}$ so that the linear Poincaré flow is strictly $\mathcal{J}_0$-monotonous on Γ.

A singularity σ is *sectionally hyperbolic with index* $\operatorname{ind}(\sigma)$ if σ is a hyperbolic equilibrium point of X (i.e. $X(\sigma) = \vec{0}$ and $\operatorname{sp}(DX(\sigma)) \cap i\mathbb{R} = \emptyset$) with stable direction E^s_σ having dimension $\operatorname{ind}(\sigma)$ and a central direction E^c_σ such that $T_\sigma M = E^s_\sigma \oplus E^c_\sigma$ is a $DX_t(\sigma)$-invariant splitting, E^s_σ is uniformly contracted and E^c_σ is sectionally expanded by the action of $DX_t(\sigma)$.

As before, the condition of $\mathcal{J}$-monotonicity for the Linear Poincare Flow can be expressed using only the vector field X and its space derivative DX as follows.

Proposition 1.4. *A $\mathcal{J}$-non-negative vector field X on a forward invariant region U has a Linear Poincare Flow which is (strictly) $\mathcal{J}$-monotone if, and only if, the operator $\hat{J}_x := DX(x)^* \cdot \Pi_x^* J \Pi_x + \Pi_x^* J \Pi_x \cdot DX(x)$ is a (positive) non-negative self-adjoint operator, that is, all eigenvalues are (positive) non-negative, for each $x \in U$ such that $X(x) \neq \vec{0}$.*

The conditions above are again consequence of the corresponding results for linear multiplicative cocycles over flows, as explained in Section 4.

1.3. **Examples.** With the equivalence provided by Theorems A and D and Corollaries B and C, we have plenty of examples illustrating this theory: singular-hyperbolic attractors and attracting sets from [27, 24, 25] and even for higher dimensions from [8], as well as the classical Lorenz attractor from the Lorenz ODE system [39]; and all hyperbolic attractors of C^1 flows.

The following examples illustrate the fact that the change of coordinates to adapt the quadratic forms as explained in Section 2 is important in applications.

Example 1. Given a diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the 2-torus, let $X_t : M \to M$ be a suspension flow with roof function $r : M \to [r_0, r_1]$ over the base transformation f, where $0 < r_0 < r_1$ are fixed, as follows.

We define $M := \{(x, y) \in \mathbb{T}^2 \times [0, +\infty) : 0 \le y < r(x)\}$. For $x = x_0 \in \mathbb{T}^2$ we denote by x_n the nth iterate $f^n(x_0)$ for $n \ge 0$ and by $S_n\varphi(x_0) = S_n^f \varphi(x_0) = \sum_{j=0}^{n-1} \varphi(x_j)$ the nth-ergodic sum, for $n \ge 1$ and for any given real function $\varphi : \mathbb{T}^2 \to \mathbb{R}$ in what follows. Then for each pair $(x_0, s_0) \in M$ and $t > 0$ there exists a unique $n \ge 1$ such that $S_n r(x_0) \le s_0 + t < S_{n+1} r(x_0)$ and we define

$$X_t(x_0, s_0) = \big(x_n, s_0 + t - S_n r(x_0)\big).$$

We note that the vector field corresponding to this suspension flow is the constant vector field $X = (0, 1)$. We observe that the space M becomes a compact manifold if we identify $(x, r(x))$ with $(f(x), 0)$; see e.g. [29].

Hence, if we are given a field of quadratic forms $\mathcal{J}$ on M and do not change coordinates accordingly, we obtain $DX \equiv 0$ and so the relation provided by Proposition 1.3 will not be fulfilled, because $\tilde{J}_x - \delta(x) J = -\delta(x) J$ is not positive definite for any choice of δ.

Remark 1.5. Moreover, if a flow X_t is such that the infinitesimal generator X is constant in the ambient space is $\mathcal{J}$-separated, then strict $\mathcal{J}$-separation implies that $-\delta(x)J$ is positive definite for all $x \in U$, and so δ is the null function on the trapping region.

Example 2. Now consider the same example as above but now f is an Anosov diffeomorphism of $\mathbb{T}^2$ with the hyperbolic splitting $E^s \oplus E^u$ defined at every point. Then the semiflow will be partially hyperbolic with splitting $E^s \oplus (E^X \oplus E^u)$ where E^X is the one-dimensional bundle spanned by the flow direction: $E^X_{(x,s)} = \mathbb{R} \cdot X(x,s), (x,s) \in M$.

Hence, Theorem A ensures the existence of a field $\mathcal{J}$ of quadratic forms such that X^t is strictly $\mathcal{J}$-separated.

Comparing with the observation at the end of Example 1, this demands a change of coordinates and, in those coordinates, the vector field X will no longer be a constant vector field.

Example 3. Now we present a suspension flow whose base map has a dominated splitting but the flow does not admit any dominated splitting.

Let $f : \mathbb{T}^4 \times \mathbb{T}^4$ be the diffeomorphism described in [9] which admits a continuous dominated splitting $E^{cs} \oplus E^{cu}$ on $\mathbb{T}^4$, but does not admit any hyperbolic (uniformly contracting or expanding) sub-bundle. There are hyperbolic fixed points of f satisfying, see Figure 1:

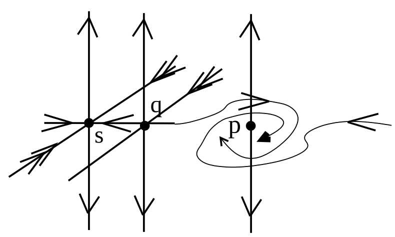

FIGURE 1. Saddles with real and complex eigenvalues.

- $\dim E^u(p) = 2 = \dim E^s(p)$ and there exists no invariant one-dimensional sub-bundle of $E^u(p)$;
- $\dim E^u(\widetilde{p}) = 2 = \dim E^s(\widetilde{p})$ and there exists no invariant one-dimensional sub-bundle of $E^s(\widetilde{p})$;
- $\dim E^s(\widetilde{q}) = 3$ and $\dim E^u(q) = 3$.

Hence, the suspension semiflow of f with constant roof function 1 does not admit any dominated splitting. In fact, the natural invariant splitting $E^{cs} \oplus E^X \oplus E^{cu}$ is the continuous invariant splitting over $\mathbb{T}^4 \times [0,1]$ with bundles of least dimension, and is not dominated since at the point p the flow direction $E^X(p)$ dominates the $E^{cs}(p) = E^s(p)$ direction, but at the point q this domination is impossible.

1.4. **Organization of the text.** We study $\mathcal{J}$-separated linear multiplicative cocycles over flows in Section 2.2, where we prove the main results whose specialization for the derivative cocycle of a smooth flow provide the main theorems, including Proposition 1.3. We then consider the case of the derivative cocycle and prove Theorem A and Corollaries B and C

in Section 3. We turn to study sectional-expanding attracting sets in Section 4, where we prove Theorem D and Proposition 1.4.

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2. Some properties of quadratic forms and $\mathcal{J}$-separated cocycles

The assumption that M is a compact manifold enables us to globally define an inner product in E with respect to which we can find the an orthonormal basis associated to $\mathcal{J}_x$ for each x, as follows. Fixing an orthonormal basis on E_x we can define the linear operator

$$J_x : E_x \to E_x \quad \text{such that} \quad \mathcal{J}_x(v) =< J_x v, v > \quad \text{for all} \quad v \in T_x M,$$

where $<,>=<,>_x$ is the inner product at E_x. Since we can always replace J_x by $(J_x + J_x^*)/2$ without changing the last identity, where J_x^* is the adjoint of J_x with respect to $<,>$, we can assume that J_x is self-adjoint without loss of generality. Hence, we represent $\mathcal{J}(v)$ by a non-degenerate symmetric bilinear form $< J_x v, v >_x$.

2.1. Adapted coordinates for the quadratic form.

Now we use Lagrange's method to diagonalize this bilinear form, obtaining a base $\{u_1, \dots, u_n\}$ of E_x such that

$$\mathcal{J}_x(\sum_i \alpha_i u_i) = \sum_{i=1}^{q} -\lambda_i \alpha_i^2 + \sum_{j=q+1}^{n} \lambda_j \alpha_j^2, \quad (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n.$$

Replacing each element of this base according to $v_i = |\lambda_i|^{1/2} u_i$ we deduce that

$$\mathcal{J}_x(\sum_i \alpha_i v_i) = \sum_{i=1}^{q} -\alpha_i^2 + \sum_{j=q+1}^{n} \alpha_j^2, \quad (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n.$$

Finally, we can redefine $<,>$ so that the base $\{v_1, \dots, v_n\}$ is orthonormal. This can be done smoothly in a neighborhood of x in M since we are assuming that the quadratic forms are non-degenerate; the reader can check the method of Lagrange in a standard Linear Algebra textbook and observe that the steps can be performed with small perturbations, for instance in [21].

In this adapted inner product we have that J_x has entries from $\{-1, 0, 1\}$ only, $J_x^* = J_x$ and also that $J_x^2 = J_x$.

Having fixed the orthonormal frame as above, the *standard negative subspace* at x is the one spanned by $v_1, \dots, v_q$ and the *standard positive subspace* at x is the one spanned $v_{q+1}, \dots, v_n$.

2.1.1. *$\mathcal{J}$-symmetrical matrixes and $\mathcal{J}$-selfadjoint operators.*

The symmetrical bilinear form defined by $(v, w) = \langle J_x v, w \rangle$, $v, w \in E_x$ for $x \in M$ endows E_x with a pseudo-Euclidean structure. Since $\mathcal{J}_x$ is non-degenerate, then the form $(\cdot, \cdot)$ is likewise non-degenerate and many properties of inner products are shared with symmetrical non-degenerate bilinear forms. We state some of them below.

Proposition 2.1. *Let $(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a real symmetric non-degenerate bilinear form on the real finite dimensional vector space V.*

1. *E is a subspace of V for which $(\cdot,\cdot)$ is non-degenerate if, and only if, $V = E \oplus E^\perp$.*
   We recall that $E^\perp := \{v \in V : (v,w) = 0 \quad \text{for all} \quad w \in E\}$, the pseudo-orthogonal space of E, is defined using the bilinear form.
2. *Every base $\{v_1, \dots, v_n\}$ of V can be orthogonalized by the usual Gram-Schmidt process of Euclidean spaces, that is, there are linear combinations of the basis vectors $\{w_1, \dots, w_n\}$ such that they form a basis of V and $(w_i, w_j) = 0$ for $i \neq j$. Then this last base can be pseudo-normalized: letting $u_i = |(w_i, w_i)|^{-1/2} w_i$ we get $(u_i, u_j) = \pm\delta_{ij}, i,j = 1,\dots,n$.*
3. *There exists a maximal dimension p for a subspace P_+ of $\mathcal{J}$-positive vectors and a maximal dimension q for a subspace P_- of $\mathcal{J}$-negative vectors; we have $p+q = \dim V$ and q is known as the* index *of $\mathcal{J}$.*
4. *For every linear map $L : V \to \mathbb{R}$ there exists a unique $v \in V$ such that $L(w) = (v,w)$ for each $w \in V$.*
5. *For each $L : V \to V$ linear there exists a unique linear operator $L^+ : V \to V$ (the pseudo-adjoint) such that $(L(v), w) = (v, L^+(w))$ for every $v, w \in V$.*
6. *Every pseudo-self-adjoint $L : V \to V$, that is, such that $L = L^+$, satisfies*
   (a) *eigenspaces corresponding to distinct eigenvalues are pseudo-orthogonal;*
   (b) *if a subspace E is L-invariant, then $E^\perp$ is also L-invariant.*

The proofs are rather standard and can be found in [21].

2.2. **Properties of $\mathcal{J}$-separated cocycles.**

In what follows we usually drop the subscript indicating the point where $\mathcal{J}$ is calculated to avoid heavy notation, since the base point is clear from the context.

2.2.1. *$\mathcal{J}$-separated linear maps.*

The following simple result will be very useful in what follows.

Lemma 2.2. *Let V be a real finite dimensional vector space endowed with a non-positive definite and non-degenerate quadratic form $\mathcal{J} : V \to \mathbb{R}$.*

If a symmetric bilinear form $F : V \times V \to \mathbb{R}$ is non-negative on C_0 then

$$r_+ = \inf_{v \in C_+} \frac{F(v,v)}{\langle Jv, v\rangle} \geq \sup_{u \in C_-} \frac{F(u,u)}{\langle Ju, u\rangle} = r_-$$

and for every r in $[r_-, r_+]$ we have $F(v,v) \geq r\langle Jv, v\rangle$ for each vector v.

In addition, if $F(\cdot,\cdot)$ is positive on $C_0 \setminus \{\vec{0}\}$, then $r_- < r_+$ and $F(v,v) > r\langle Jv, v\rangle$ for all vectors v and $r \in (r_-, r_+)$.

Proof. This can be found in [46] and also in [31]. We present the simple proof here for completeness.

Let us assume that the F is non-negative on C_0 and argue by contradiction: we also assume that

$$\inf_{v\in C_+}\frac{F(v,v)}{\langle Jv,v\rangle} < \sup_{u\in C_-}\frac{F(u,u)}{\langle Ju,u\rangle}. \tag{2.1}$$

Hence we can find $v_0 \in C+$ and $u_0 \in C_-$ with $\mathcal{J}(v_0) = 1$ and $\mathcal{J}(u_0) = -1$ such that $F(v_0, v_0) + F(u_0, u_0) < 0$. We can also find an angle α such that both linear combinations

$$v = v_0 \cos\alpha + u_0 \sin\alpha \quad \text{and} \quad w = -v_0 \sin\alpha + u_0 \cos\alpha$$

belong to C_0. Then we must have $F(v, v) \geq 0$ and $F(w, w) \geq 0$, but we also have

$$\begin{aligned} F(v,v) + F(w,w) &= \cos^2\alpha \cdot F(v_0,v_0) + + \sin 2\alpha \cdot F(u_0,v_0) + \sin^2\alpha \cdot F(u_0,u_0) \\ &\qquad + \sin^2\alpha \cdot F(v_0,v_0) - \sin 2\alpha \cdot F(u_0,v_0) + \cos^2\alpha \cdot F(u_0,u_0) \\ &= F(v_0,v_0) + F(u_0,u_0) < 0 \end{aligned}$$

and this contradiction shows that the opposite of (2.1) must be true.

Analogously, if F is positive on $C_0 \setminus \{\vec{0}\}$, then we can argue in the same way: we assume that (2.1) is true with $\leq$ in the place of $<$; we obtain $F(v_0, v_0) + F(u_0, u_0) \leq 0$ and then construct v, w such that $F(v, v) + F(w, w) > 0$; and, finally, we show that $F(v, v) + F(w, w) = F(v_0, v_0) + F(u_0, u_0) \leq 0$ to arrive again at a contradiction. □

Remark 2.3. Lemma 2.2 shows that if $F(v, w) = \langle \tilde{J}v, w\rangle$ for some self-adjoint operator $\tilde{J}$ and $F(v, v) \geq 0$ for all v such that $\langle Jv, v\rangle = 0$, then we can find $a \in \mathbb{R}$ such that $\tilde{J} \geq aJ$. This means precisely that $\langle \tilde{J}v, v\rangle \geq a\langle Jv, v\rangle$ for all v.

If, in addition, we have $F(v, v) > 0$ for all v such that $\langle Jv, v\rangle = 0$, then we obtain a strict inequality $\tilde{J} > aJ$ for some $a \in \mathbb{R}$ since the infimum in the statement of Lemma 2.2 is strictly bigger than the supremum.

The (longer) proofs of the following results can be found in [46] or in [31]; see also [47].

Proposition 2.4. *Let $L : V \to V$ be a $\mathcal{J}$-separated linear operator. Then*

1. *L can be uniquely represented by $L = RU$, where U is a $\mathcal{J}$-isometry (i.e. $\mathcal{J}(U(v)) = \mathcal{J}(v), v \in V$) and R is $\mathcal{J}$-symmetric (or $\mathcal{J}$-pseudo-adjoint; see Proposition 2.1) with positive spectrum.*
2. *the operator R can be diagonalized by a $\mathcal{J}$-isometry. Moreover the eigenvalues of R satisfy*
$$0 < r_-^q \leq \cdots \leq r_-^1 = r_- \leq r_+ = r_1^+ \leq \cdots \leq r_+^p.$$
3. *the operator L is (strictly) $\mathcal{J}$-monotonous if, and only if, $r_- \leq (<)1$ and $r_+ \geq (>)1$.*

For a $\mathcal{J}$-separated operator $L : V \to V$ and a d-dimensional subspace $F_+ \subset C_+$, the subspaces F_+ and $L(F_+) \subset C_+$ have an inner product given by $\mathcal{J}$. Thus both subspaces are endowed with volume elements. Let $\alpha_d(L; F_+)$ be the rate of expansion of volume of $L\mid_{F_+}$ and $\sigma_d(L)$ be the infimum of $\alpha_d(L; F_+)$ over all d-dimensional subspaces F_+ of C_+.

Proposition 2.5. *We have $\sigma_d(L) = r_+^1 \cdots r_+^d$, where r_+^i are given by Proposition 2.4(2). Moreover, if L_1, L_2 are $\mathcal{J}$-separated, then $\sigma_d(L_1L_2) \geq \sigma_d(L_1)\sigma_d(L_2)$.*

The following corollary is very useful.

Corollary 2.6. *For $\mathcal{J}$-separated operators $L_1, L_2 : V \to V$ we have*

$$r_+^1(L_1L_2) \geq r_+^1(L_1)r_+^1(L_2) \quad \text{and} \quad r_-^1(L_1L_2) \leq r_-^1(L_1)r_-^1(L_2).$$

Moreover, if the operators are strictly $\mathcal{J}$-separated, then the inequalities are strict.

2.3. **$\mathcal{J}$-separated linear cocycles over flows.** The results in the previous subsection provide the following characterization of $\mathcal{J}$-separated cocycles $A_t(x)$ over a flow X_t in terms of the infinitesimal generator $D(x)$ of $A_t(x)$; see (2.2). The following statement is more precise than Proposition 1.3.

Let $A_t(x)$ a linear multiplicative cocycles over a flow X_t. We define the infinitesimal generator of $A_t(x)$ by

$$D(x) := \lim_{t\to 0} \frac{A_t(x) - Id}{t}. \tag{2.2}$$

Theorem 2.7. *Let X_t be a flow defined on a positive invariant subset U, $A_t(x)$ a cocycle over X_t on U and $D(x)$ its infinitesimal generator. Then*

1. *$\partial_t \mathcal{J}(A_t(x)v) = \langle \tilde{J}_{X_t(x)} A_t(x)v, A_t(x)v\rangle$ for all $v \in E_x$ and $x \in U$, where*
$$\tilde{J}_x := J \cdot D(x) + D(x)^* \cdot J \tag{2.3}$$
and $D(x)^*$ denotes the adjoint of the linear map $D(x) : E_x \to E_x$ with respect to the adapted inner product at x;
2. *the cocycle $A_t(x)$ is $\mathcal{J}$-separated if, and only if, there exists a neighborhood V of Λ, $V \subset U$ and a function $\delta : V \to \mathbb{R}$ such that*
$$\tilde{\mathcal{J}}_x \geq \delta(x)\mathcal{J} \quad \text{for all} \quad x \in V. \tag{2.4}$$
In particular we get $\partial_t \mathcal{J}(A_t(x)v) \geq \delta(X_t(x))\mathcal{J}(A_t(x)v)$, $v \in E_x, x \in V, t \geq 0$;
3. *if the inequalities in the previous item are strict, then the cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated. Reciprocally, if $A_t(x)$ is strictly $\mathcal{J}$-separated, then there exists an equivalent family $\mathcal{J}_0$ of forms on V satisfying the strict inequalities of item (2).*
4. *Define the function*
$$\Delta(x, t) := \int_0^t \delta(X_s(x))\, ds. \tag{2.5}$$
For a $\mathcal{J}$-separated cocycle $A_t(x)$, we have
   (a) *$\mathcal{J}(A_t(x)v) \geq \mathcal{J}(v) \exp \Delta(x, t)$ for all $v \in C_+(x), t > 0$;*
   (b) *$|\mathcal{J}(A_t(x)w)| \leq |\mathcal{J}(w)| \exp \Delta(x, t)$ for all $w \in C_-(x)$ such that $x = X^{-t}(x_0)$ and $w = A_{-t}(x_0)w_0$ for some $w_0 \in C_-(x_0)$ and $t \in [0, t_0]$ satisfying $x_0 \in U$, $t_0 > 0$ and $X_{-t}(x_0) \in U$ for all $0 \leq t \leq t_0$.*

(5) *if $A_t(x)$ is $\mathcal{J}$-separated and $x \in \Lambda(U), v \in C_+(x)$ and $w \in C_-(x)$ are non-zero vectors, then for every $t > 0$ such that $A_s(x)w \in C_-(X_s(x))$ for all $0 < s < t$*

$$\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)} \leq \frac{|\mathcal{J}(w)|}{\mathcal{J}(v)} \exp\big(2\Delta(x,t)\big). \tag{2.6}$$

Remark 2.8. If $\delta(x) = 0$, then $\tilde{\mathcal{J}}_x$ is positive semidefinite operator. But for $\delta(x) \neq 0$ the symmetric operator $\tilde{\mathcal{J}}_x$ might be an indefinite quadratic form.

Remark 2.9. The necessary condition in item (3) of Theorem 2.7 is proved in Section 2.5 after Theorem 2.16 and Proposition 2.21.

Remark 2.10. Complementing Remark 1.2, the necessary and sufficient condition in items (2-3) of Theorem 2.7, for (strict) $\mathcal{J}$-separation, shows that a cocycle $A_t(x)$ is (strictly) $\mathcal{J}$-separated if, and only if, its inverse $A_{-t}(x)$ is (strictly) $(-\mathcal{J})$-separated.

Proof of Theorem 2.7. The map $\psi(t,v) := \langle JA_t(x)v, A_t(x)v\rangle$ is smooth and for $v \in E_x$ satisfies

$$\begin{aligned}
\partial_t\psi(t,v) &= \langle \big(J \cdot D(X_t(x))\big) A_t(x)v, A_t(x)v\rangle \\
&\quad + \langle J \cdot A_t(x)v, D(X_t(x))A_t(x)v\rangle \\
&= \langle \big(J \cdot D(X_t(x)) + D(X_t(x))^* \cdot J\big) A_t(x)v, A_t(x)v\rangle,
\end{aligned}$$

where we have used the fact that the cocycle has an infinitesimal generator $D(x)$: we have the relation

$$\partial_t A_t(x)v = D(X_t(x)) \cdot A_t(x)v \quad \text{for all } t \in \mathbb{R}, x \in M \text{ and } v \in E_x. \tag{2.7}$$

This is because we have the linear variation equation: $A_t(x)$ is the solution of the following non-autonomous linear equation

$$\begin{cases} \dot{Y} = D(X_t(x))Y \\ Y(0) = Id \end{cases}. \tag{2.8}$$

We note that the argument does not change for $x = \sigma$ an equilibrium point of X_t.

This proves the first item of the statement of the theorem.

We observe that the independence of J from $X_t(x)$ is a consequence of the choice of adapted coordinates and inner product, since in this setting the operator J is fixed. However, in general, this demands the rewriting of the cocycle in the coordinate system adapted to $\mathcal{J}$.

For the second item, let us assume that $A_t(x)$ is $\mathcal{J}$-separated on U. Then, by definition, if we fix $x \in U$

$$\langle JA_t(x)v, A_t(x)v\rangle > 0 \quad \text{for all } t > 0 \text{ and all } v \in E_x \text{ such that } \langle Jv, v\rangle > 0. \tag{2.9}$$

We also note that, by continuity, we have $\langle JA_t(x)v, A_t(x)v\rangle \geq 0$ for all v such that $\langle Jv, v\rangle = 0$. Indeed, for any given $t > 0$ and $v \in C_0$ we can find $w \in C_+$ such that $v + w \in C_+$. Then we have $\langle JA_t(x)(v + \lambda w), A_t(x)(v + \lambda w)\rangle > 0$ for all $\lambda > 0$, which proves the claim letting λ tend to 0.

The map $\psi(t,v)$ satisfies $\psi(0,v)=0\le\psi(t,v)$ for all $t>0$ and $v\in C_0(x)$, hence from the first item already proved

$$0\le\partial_t\psi(t,v)\mid_{t=0}=\langle\big(J\cdot D(x)+D(x)^*\cdot J\big)v,v\rangle.$$

According to Lemma 2.2 (cf. also Remark 2.3) there exists $\delta(x)\in\mathbb{R}$ such that (2.4) is true and this, in turn, implies that $\partial_t\mathfrak{J}(A_t(x)v)\ge\delta(x)\mathfrak{J}(A_t(x)v)$, for all $v\in E_x, x\in U, t\ge 0$. This completes the proof of necessity in the second item.

To see that this is a sufficient condition for $\mathfrak{J}$-separation, let $\tilde{\mathfrak{J}}_x\ge\delta(x)\mathfrak{J}$, for some function $\delta:U\to\mathbb{R}$. Then, for all $v\in E_x$ such that $\langle Jv,v\rangle>0$, since $\partial_t\mathfrak{J}(A_t(x)v)\ge\delta(X_t(x))\mathfrak{J}(A_t(x)v)$, we obtain

$$|\mathfrak{J}(A_t(x)v)|\ge|\mathfrak{J}(v)|\exp\left(\int_0^t\delta(X_s(x))\,ds\right)=|\mathfrak{J}(v)|\exp\Delta(x,t)>0,\quad t\ge 0, \tag{2.10}$$

and $\mathfrak{J}(A_t(x)v)>0$ for all $t>0$ by continuity. This shows that $A_t(x)$ is $\mathfrak{J}$-separated. This completes the proof of the second item in the statement of the theorem.

For the third item, we only prove the first statement and leave the longer proof of the second statement for Section 2.5 in Proposition 2.21. If $\tilde{J}_x>\delta(x)J$ for all $x\in U$, then for $t>0$ we obtain (2.10) with strict inequalities for $v\in C_0(x)$, hence $\mathfrak{J}(A_t(x)v)>0$ for $t>0$. So $A_t(x)$ is strictly $\mathfrak{J}$-separated.

For the fourth item: subitem (a) is just item (2) with the observation that a positive vector remains positive under the action of the cocycle for positive t, thus we get (2.10) without the modulus signs. As for subitem (b), for each $x_0\in U$ and $t_0>0$ such that $X_{-t}(x_0)\in U$ for all $0\le t\le t_0$, fixing $x=X_{-t}(x_0)$ and $w=A_{-t}(x_0)w_0$ for some $w_0\in C_-(x_0)$ and $t\in[0,t_0]$, we get from items (1) and (2) already proved

$$\begin{aligned}&\partial_t\mathfrak{J}(A_t(x)w)\ge\delta(X_t(x))\mathfrak{J}(A_t(x)w)\quad\text{and}\quad\mathfrak{J}(A_t(x)w)<0,\\ \text{so}\quad&\frac{\partial_t\mathfrak{J}(A_t(x)w)}{\mathfrak{J}(A_t(x)w)}\le\delta(X_t(x)).\end{aligned}$$

Integrating this inequality from 0 to t we obtain

$$\log\left|\frac{\mathfrak{J}(A_t(x)w)}{\mathfrak{J}(w)}\right|\le\int_0^t\delta(X^s(x))\,ds\quad\text{thus}\quad|\mathfrak{J}(A_t(x)w)|\le|\mathfrak{J}(w)|\exp\Delta(x,t). \tag{2.11}$$

For the last item, we calculate the derivative of the ratio of the forms and use the previous results as follows

$$\begin{aligned}\partial_t\left(\frac{|\mathfrak{J}(A_t(x)w)|}{\mathfrak{J}(A_t(x)v)}\right)&=-\frac{\langle\tilde{J}_{X_t(x)}A_t(x)w,A_t(x)w\rangle}{\mathfrak{J}(A_t(x)v)}-\frac{\mathfrak{J}(A_t(x)w)}{\mathfrak{J}(A_t(x)v)}\cdot\frac{\langle\tilde{J}_{X_t(x)}A_t(x)v,A_t(x)v\rangle}{\mathfrak{J}(A_t(x)v)}\\&\le-\delta(X_t(x))\frac{\mathfrak{J}(A_t(x)w)}{\mathfrak{J}(A_t(x)v)}-\frac{\mathfrak{J}(A_t(x)w)}{\mathfrak{J}(A_t(x)v)}\cdot\delta(X_t(x))\\&=2\delta(X_t(x))\cdot\frac{|\mathfrak{J}(A_t(x)w)|}{\mathfrak{J}(A_t(x)v)}\end{aligned}$$

and the result is obtained by integrating this equation from 0 to t. The proof is complete. $\square$

In Section 2.6 we show that the asymptotic behavior of the function

$$\Delta(x,t) := \int_0^t \delta(X_s(x))\,ds$$

as t grows to $\pm\infty$ defines the type of partial hyperbolic splitting exhibited by a strictly $\mathcal{J}$-separated cocycle.

In this way we have a condition ensuring partial and uniform hyperbolicity of an invariant subset involving only the vector field and its spatial derivative map.

Remark 2.11. Considering

$$a := \inf\{\langle \tilde{J}_x v, v\rangle / \langle Jv, v\rangle : v \in C_+\} \text{ and } b := \sup\{\langle \tilde{J}_x v, v\rangle / \langle Jv, v\rangle : v \in C_-\},$$

the arguments above show that $\delta(x)$ can be taken in $[a,b]$ in the $\mathcal{J}$-separated case; and in the interior of the above interval in the strictly $\mathcal{J}$-separated case. Thus we can take $\delta(x)$ as a continuous function of the point $x \in U$.

Remark 2.12. The inequality (2.10) shows that δ is a measure of the "minimal instantaneous expansion rate" of $|\mathcal{J} \circ A_t(x)|$ on positive vectors; (2.11) shows that δ is also a "maximal instantaneous expansion rate" of $|\mathcal{J} \circ A_t(x)|$ on negative vectors; and the last inequality shows in addition that δ is also a bound for the "instantaneous variation of the ratio" between $|\mathcal{J} \circ A_t(x)|$ on negative and positive vectors.

2.4. **Strict $\mathcal{J}$-separated cocycles and domination.** We assume from now on that a family $A_t(x)$ of linear multiplicative cocycles on a vector bundle E_U over the flow X_t on a trapping region $U \subset M$ has been given, together with a field $\mathcal{J}$ of non-degenerate quadratic forms on E_U with constant index $q < \dim E_U$.

Theorem 2.13. *The cocyle $A_t(x)$ is strictly $\mathcal{J}$-separated if, and only if, E_U admits a dominated splitting $F_- \oplus F_+$ with respect to $A_t(x)$ on the maximal invariant subset Λ of U, with constant dimensions* $\dim F_- = q, \dim F_+ = p, \dim M = p + q$.

Moreover the properties stated in Theorem 2.13 are robust: they hold for all nearby cocycles on E_U over all flows close enough to X_t; see Section 2.5.

We now start the proof of Theorem 2.13. We construct a decomposition of the tangent space over Λ into a direct sum of invariant subspaces and then we prove that this is a dominated splitting.

2.4.1. *The cones are contracted.* To obtain the invariant subspaces, we show that the action of $A_t(x)$ on the family of all p-dimensional spaces inside the positive cones is a contraction in the appropriate distance. For that we use a result from [46].

Let us fix $C_+ = C_+(x)$ for some $x \in \Lambda$ and consider the family $G_p(C_+)$ of all p-subspaces of C_+, where $p = n - q$. This manifold can be identified with the family of all $q \times p$ matrices T with real entries such that $T^*T < I_p$, where I_p is the $p \times p$ identity matrix and $<$ indicates that for the standard inner product in $\mathbb{R}^p$ we have $< T^*Tu, u > < < u, u >$, for all $u \in \mathbb{R}^p$.

A $\mathcal{J}$-separated operator naturally sends $G_p(C_+)$ inside itself. This operation is a contraction.

Theorem 2.14. *There exists a distance* dist *on $G_p(C_+)$ so that $G_p(C_+)$ becomes a complete metric space and, if $L : V \to V$ is $\mathcal{J}$-separated and $T_1, T_2 \in G_p(C_+)$, then*

$$\operatorname{dist}(L(T_1), L(T_2)) \le \frac{r_-}{r_+} \operatorname{dist}(T_1, T_2),$$

where $r_\pm$ are given by Proposition 2.4.

Proof. See [46, Theorem 1.6]. □

2.4.2. *Invariant directions.* Now we consider a pair $C_-(x)$ and $C_-(X_{-t}(x))$ of positive cones, for some fixed $t > 0$ and $x \in \Lambda$, together with the linear isomorphism $A_{-t}(x) : E_x \to E_{X_{-t}(x)}$. We note that the assumption of strict $\mathcal{J}$-separation ensures that $A_{-t}(x) \mid C_-(x) : C_-(x) \to C_-(X_{-t}(x))$. We have in fact

$$\overline{A_{-t}(x) \cdot C_-(x)} \subset C_-(X_{-t}(x)). \tag{2.12}$$

Moreover, by Theorem 2.14 we have that the diameter of $A_{-nt}(x) \cdot C_-(X_{nt}(x))$ decreases exponentially fast when n grows. Hence there exists a unique element $F_-(x) \in G_q(C_-(x))$ in the intersection of all these cones. Analogous results hold for the positive cone with respect to the action of $A_t(x)$. It is easy to see that

$$A_t(x) \cdot F_\pm(x) = F_\pm(X_t(x)), \quad x \in \Lambda. \tag{2.13}$$

Moreover, since the strict inclusion (2.12) holds for whatever $t > 0$ we fix, then we see that the subspaces $F_\pm$ do not depend on the chosen $t > 0$.

2.4.3. *Domination.* The contraction property on C_+ for $A_t(x)$ and on C_- for $A_{-t}(x)$, any $t > 0$, implies domination directly. Indeed, let us fix $t > 0$ in what follows and consider the norm $|\cdot|$ induced on E_x for each $x \in U$ by

$$|v| := \sqrt{\mathcal{J}(v_-)^2 + \mathcal{J}(v_+)^2} \quad \text{where} \quad v = v_- + v_+, v_\pm \in F_\pm(x).$$

Now, according to Lemma 2.2 together with Proposition 2.4 we have that, for each $x \in \overline{X_t(U)}$ and every pair of unit vectors $u \in F_-(x)$ and $v \in F_+(x)$

$$\frac{|A_t(x)u|}{|A_t(x)v|} \le \frac{r_-^t(x)}{r_+^t(x)} \le \omega_t := \sup_{z \in \overline{X_t(U)}} \frac{r_-^t(z)}{r_+^t(z)} < 1,$$

where $r_\pm^t(x)$ represent the values $r_\pm$ shown to exist by Lemma 2.2 with respect to the strictly $\mathcal{J}$-separated linear map $A_t(x)$. The value of ω_t is strictly smaller than 1 by continuity of the functions $r_\pm$ on the compact subset $\overline{X_t(U)}$.

Now we use the following well-known lemma.

Lemma 2.15. *Let X_t be a C^1 flow and Λ a compact invariant set for X_t admitting a continuous invariant splitting $T_\Lambda M = F_- \oplus F_+$. Then this splitting is dominated if, and only if, there exists a riemannian metric on Λ inducing a norm such that*

$$\lim_{t \to +\infty} \|A_t(x) \mid_{F_-(x)} \| \cdot \|A_{-t}(X_t(x)) \mid_{F_+(X_t(x))} \| = 0,$$

for all $x \in \Lambda$.

This shows that for the cocycle $A_t(x)$ the splitting $E_\Lambda = F_- \oplus F_+$ is dominated, since the above argument does not depend on the choice of $t > 0$ and implies that

$$\lim_{t\to+\infty} \frac{|A_{nt}(x)u|}{|A_{nt}(x)v|} = 0,$$

and we conclude that

$$\lim_{t\to+\infty} \frac{|A_t(x)u|}{|A_t(x)v|} = 0, \quad x \in \Lambda, u \in F_-(x), v \in F_+(x).$$

2.4.4. *Continuity of the splitting.* The continuity of the subbundles $F_\pm$ over Λ is a consequence of domination together with the observation that the dimensions of $F_\pm(x)$ do not depend on $x \in \Lambda$; see for example [6, Appendix B]. Moreover, since we are assuming that $A_t(x)$ is smooth, i.e. the cocycle admits an infinitesimal generator, then the $A_t(x)$-invariance ensures that the subbundles $F_\pm(x)$ can be differentiated along the orbits of the flow.

This completes the proof that strict $\mathcal{J}$-separation implies a dominated splitting, as stated in Theorem 2.13.

2.5. **Domination implies strict $\mathcal{J}$-separation.** Now we start the proof of the converse of Theorem 2.13 by showing that, given a dominated decomposition of a vector bundle over a compact invariant subset Λ of the base, for a C^1 vector field X on a trapping region U, there exists a smooth family of quadratic forms $\mathcal{J}$ for which Y is strictly $\mathcal{J}$-separated on E_V over a neighborhood V of Λ for each vector field Y sufficiently C^1 close to X and every cocycle close enough to A_t.

We define a distance between smooth cocycles as follows. If $D_A(x), D_B(x) : E_x \to E_x$ are the infinitesimal generators of the cocycles $A_t(x), B_t(x)$ over the flow of X and Y respectively, then we can recover the cocycles through the non-autonomous ordinary differential equation (2.8). We then define the *distance d between the cocycles A_t and B_t* to be

$$d\big((A_t)_t, (B_t)_t\big) := \sup_{x\in M} \|D_A(x) - D_B(x)\|,$$

where $\|\cdot\|$ is a norm on the vector bundle E. We always assume that we are given a Riemannian inner product in E which induces the norm $\|\cdot\|$.

As before, let $\Lambda = \Lambda(U)$ be a maximal positively invariant subset for a C^1 vector field X endowed with a linear multiplicative cocycle $A_t(x)$ defined on a vector bundle over U.

Theorem 2.16. *Suppose that Λ has a dominated splitting $E_\Lambda = F_- \oplus F_+$. Then there exists a C^1 field of quadratic forms $\mathcal{J}$ on a neighborhood $V \subset U$ of Λ, a C^1-neighborhood $\mathcal{V}$ of X and a C^0-neighborhood $\mathcal{W}$ of $A_t(x)$ such that $B_t(x)$ is strictly $\mathcal{J}$-separated on V with respect to $Y \in \mathcal{V}$ and $B \in \mathcal{W}$.*

To prove this result, all we need to do is to prove the following equivalent lemma.

Lemma 2.17. *Under the hypothesis of Theorem 2.16, there are a constants $\kappa_1, \kappa_2, \omega > 0$ and a function $\kappa : V \to (0, 2)$ such that, for each $x \in V$ and $t \geq 0$*

(1) $|\mathcal{J}(A_t(x)v_-)| \le \kappa_{x,t} e^{-\omega t}\mathcal{J}(A_t(x)v_+)$, $v_\pm \in F_\pm(x)$;
(2) $0 < 1-\kappa_1 \le \kappa_x \le 1+\kappa_2 < 2$ *and* $|\partial_t \kappa_{x,t}| \le \omega/100$.

Remark 2.18. This result shows, in particular, that if two families of quadratic forms $\mathcal{J}_1$, $\mathcal{J}_2$ have the same signs on the subbundles of a dominated splitting of E_Λ over a compact set Λ, then $\mathcal{J}_1$ and $\mathcal{J}_2$ are equivalent forms. Indeed, $\hat{J}\mid_{F_+(x)}$ and $\mathcal{J}\mid_{F_+(x)}$ are both positive definite quadratic forms, i.e., they define inner products. Thus they must be equivalent: there exists $C_+(x) > 0$ such that $C_+(x)^{-1}\mathcal{J}_2 \le \mathcal{J}_1 \le C_+(x)\mathcal{J}_2$ on $F_+(x)$. Analogously for the restriction to $F_-(x)$. Now we just have to take $C = \max\{C_+(x), C_-(x) : x \in \Lambda\}$ since $x \mapsto C_\pm(x)$ can be taken as continuous functions because $x \mapsto \mathcal{J}_1(x)$ and $x \mapsto \mathcal{J}_2(x)$ are also continuous.

To prove this we use the following result from [13], ensuring the existence of adapted metrics for dominated splittings over Banach bundle automorphisms and flows.

Let Λ be a compact invariant set for a C^1 vector field X and let E be a vector bundle over M.

Theorem 2.19. *Suppose that $T_\Lambda M = F_- \oplus F_+$ is a dominated splitting for a linear multiplicative cocycle $A_t(x)$ over E. There exists a neighborhood V of Λ and a Riemannian metric $<< \cdot,\cdot >>$ inducing a norm $|\cdot|$ on E_V such that there exists $\lambda > 0$ satisfying for all $t > 0$ and $x \in \Lambda$*

$$|A_t(x)|_{F_-(x)}| \cdot \left|(A_t(x)|_{F_+(x)})^{-1}\right| < e^{-\lambda t}.$$

Moreover, in the adapted metric the bundles $F_\pm$ over Λ are orthogonal.

Remark 2.20. A similar result holds for the existence of adapted metric for partially hyperbolic and for uniformly hyperbolic splittings.

We may assume, without loss of generality, that V given by Theorem 2.19 coincides with U. Now, we use the adapted riemannian metric to define the quadratic form on a smaller neighborhood of Λ inside U.

2.5.1. *Construction of the field of quadratic forms.* First we choose a continuous family of orthonormal basis (with respect to the adapted metric) $\{e_1(x), \dots, e_s(x)\}$ of $F_-(x)$ and $\{e_{s+1}(x), \dots, e_{s+c}(x)\}$ of $F_+(x)$ for $x \in \Lambda$, where $s = \dim F_-$ and $c = \dim F_+$. Then we note that $\{e_i(x)\}_{i=1}^{s+c}$ is a orthonormal basis for E_x, $x \in \Lambda$.

Secondly, we consider the following quadratic forms

$$\mathcal{J}_x(v) = \mathcal{J}_x\left(\sum_{i=1}^{s+c}\alpha_i e_i(x)\right) := |v^+|^2 - |v^-|^2 = \sum_{i=s+1}^{s+c}\alpha_i^2 - \sum_{i=1}^{s}\alpha_j^2, \quad v \in E_x, x \in V,$$

where $v^\pm \in F_\pm(x)$ are the unique projections on the subbundles such that $v = v^- + v^+$. This defines a family of quadratic forms on Λ.

We note that, since $F_- \oplus F_+$ is $A_t(x)$-invariant over Λ, and the vector field X and the flow X_t are C^1, the quadratic forms constructed above are a differentiable family along the flow direction, because $F_\pm(X_t(x)) = A_t(x) \cdot F_\pm(x)$ is a differentiable family in $t \in \mathbb{R}$ for each $x \in \Lambda$.

Clearly F_- is a $\mathcal{J}$-negative subspace and F_+ is a $\mathcal{J}$-positive subspace, which shows that the index of $\mathcal{J}$ equals s and that the forms are non-degenerate.

In addition, we have strict $\mathcal{J}$-separation over Λ. Indeed, $v = v^- + v^+ \in C_+(x) \cup C_0(x)$ for $x \in \Lambda$ means $|v^+| \geq |v^-|$ and the $A_t(x)$-invariance of $F_\pm$ ensures that $A_t(x)v = A_t(x)v^- + A_t(x)v^+$ with $A_t(x)v^\pm \in F_\pm(X_t(x))$ and $\sqrt{\mathcal{J}(A_t(x)v^+)} = |A_t(x)v^+| > e^{\lambda t}|A_t(x)v^-| = \sqrt{|\mathcal{J}(A_t(x)v^-)|}$, so that $A_t(x)v \in C_+(X_t(x))$.

We are ready to obtain the reciprocal of item 3 of Theorem 2.7.

Proposition 2.21. *If the cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated over a compact X_t-invariant subset Λ, then there exist an equivalent family $\mathcal{J}_0$ of quadratic forms and a function $\delta : \Lambda \to \mathbb{R}$ such that $\tilde{\mathcal{J}}_{0,x} > \delta(x)\mathcal{J}_0$ for all $x \in \Lambda$.*

Proof. We have already shown that a strictly $\mathcal{J}$-separated cocycle has a dominated splitting $E = F_- \oplus F_+$ in Section 2.4. Then we build the field of quadratic forms $\mathcal{J}_0$ according to the previous arguments in this section, and calculate for $v_0 \in C_0(x), v_0 = v^- + v^+$ with $v^\pm \in F_\pm(x)$ and $|v^\pm| = 1$, for a given $x \in \Lambda$ and all $t > 0$

$$\mathcal{J}_0(A_t(x)v_0) = |A_t(x)v^-|^2 \left(\frac{|A_t(x)v^+|^2}{|A_t(x)v^-|^2} - 1 \right) \geq |A_t(x)v^-|^2 \cdot (e^{2\lambda t} - 1). \tag{2.14}$$

The derivative of the right hand side above satisfies

$$2\lambda e^{2\lambda t}|A_t(x)v^-|^2 + (e^{2\lambda t} - 1)\partial_t |A_t(x)v^-|^2 \xrightarrow[t\searrow 0]{} 2\lambda.$$

Since the left hand side and the right hand side of (2.14) have the same value at $t = 0$ (we note that $\mathcal{J}_0(v_0) = 0$ by the choice of v_0), we have

$$\tilde{\mathcal{J}}_x(v_0) = \partial_t \mathcal{J}_0(A_t(x)v_0) \mid_{t=0} \geq 2\lambda > 0, \quad x \in \Lambda.$$

Thus, $\tilde{\mathcal{J}}_x(v_0) > 0$ for $\vec{0} \neq v_0 \in C_0(x)$ which implies by Lemma 2.2 that $\tilde{\mathcal{J}}_x > \delta(x)\mathcal{J}_0$ for some real function $\delta(x)$. Finally, the quadratic forms $\mathcal{J}_0$ and $\mathcal{J}$ are equivalent, so we obtain the claimed strict inequality. ☐

2.5.2. *Continuous/smooth extension to a neighborhood.* We choose a C^0 extension of the function $\mathcal{J} : E_\Lambda \to \mathbb{R}$ given by a continuous family of quadratic forms on an open neighborhood V of Λ, which we denote by $\hat{\mathcal{J}}$. This extension is performed noting that it is enough to have the restriction of $\mathcal{J}_x$ to the unit sphere $\mathbb{S}_x$ at E_x to define $\mathcal{J}_x : E_x \to \mathbb{R}$ by $v \mapsto |v|^2\mathcal{J}_x(v/|v|)$. Then we observe that $E^1_\Lambda := \{\mathbb{S}_x : x \in \Lambda\}$ is a compact subset of the submanifold $E^1 := \{\mathbb{S}_x : x \in M\}$ of E, so there exists a C^0 extension $\hat{\mathcal{J}}$ of $\mathcal{J} \mid_{E^1_\Lambda}$ to $E^1_V = \{\mathbb{S}_x : x \in V\}$. This defines a continuous family $\hat{\mathcal{J}}$ of quadratic forms on E_V, the restriction of the vector bundle E to V, which extends the continuous family $\mathcal{J}$ of quadratic forms we had over Λ.

We recall that the family $\mathcal{J}$ is differentiable along the flow direction. Thus the extension remains differentiable along the flow direction over the points of Λ.

We can also consider in the same way a C^0 approximation of $\hat{\mathcal{J}} \mid_{E^1_V}$ by a C^1 function $\bar{\mathcal{J}}$ on E^1_V. In fact, $\bar{\mathcal{J}}$ can be seen as a C^1 regularization of $\hat{\mathcal{J}}$. Hence $\bar{\mathcal{J}}$ is automatically C^1

close to $\hat{\mathcal{J}}$ over orbits of the flow on Λ, i.e., $\bar{\mathcal{J}}$ is C^1-close to $\hat{\mathcal{J}}$ on Λ. This means that, given $\varepsilon > 0$, we can find $\bar{\mathcal{J}}$ such that

- $|\hat{\mathcal{J}}_y(v) - \bar{\mathcal{J}}_y(v)| < \varepsilon$ for all $v \in E_y, y \in V$ (C^0-closeness on V);
- $|\partial_t \hat{\mathcal{J}}_{X_t(x)}(A_t(x)v) - \partial_t \bar{\mathcal{J}}_{X_t(x)}(A_t(x)v)| < \varepsilon$ for all $v \in E_x, x \in \Lambda$ and $t \in \mathbb{R}$.

In addition, we note that $\sigma_{x,t} := \mathcal{J}(A_t(x)v_\pm)e^{-2\lambda t}/\mathcal{J}(v_\pm) = 1$ for all $x \in \Lambda, \vec{0} \neq v_\pm \in F_\pm(x)$ and $t \geq 0$. Hence $\partial_t \sigma_{x,t} \equiv 0$ for $x \in \Lambda$ and

$$\kappa_{x,t} := e^{-2\lambda t} \frac{\bar{\mathcal{J}}(A_t(x)v_\pm)}{\bar{\mathcal{J}}(v_\pm)} = \frac{\bar{\mathcal{J}}(A_t(x)v_\pm)}{\mathcal{J}(A_t(x)v_\pm)} \cdot \sigma_{x,t} \cdot \frac{\mathcal{J}(v_\pm)}{\bar{\mathcal{J}}(v_\pm)}$$

depends only on the ratio between the forms $\mathcal{J}$ and $\bar{\mathcal{J}}$ at $X_t(x)$ and x, for $x \in \Lambda$. We can make this ratio as close to 1 as we need. In particular, we assume that $e^{-\lambda/2} \leq \kappa_x \leq e^{\lambda/2}$. Moreover, since $\kappa_{x,t}$ is differentiable in t, we can ensure also that $|\partial_t \kappa_{x,t}| \leq \lambda/200$ for all $x \in \Lambda$ just by choosing a smaller value for $\varepsilon > 0$ in the choice of $\bar{\mathcal{J}}$. Therefore we have the relations in the statement of Theorem 2.16 if we set $\omega = 2\lambda$.

2.5.3. *Strict separation for the extension/smooth approximation.* We now show that Y is strictly $\bar{\mathcal{J}}$-separated on V for every vector field Y in a neighborhood $\mathcal{V}$ of X and for every multiplicative cocycle $B_t(x)$ over Y which is C^0 close to $A_t(x)$.

We start by observing that $\bar{\mathcal{J}}$ is differentiable along the flow direction. Then we note that, from Proposition 2.21

$$\iota := \inf\{\widetilde{\mathcal{J}}_x - \delta(x)\mathcal{J}_x : x \in \Lambda\} > 0$$

and recall that $\widetilde{J}_x = J_x \cdot D(x) + D(x)^* \cdot J_x$. Hence, by choosing V sufficiently small around Λ, we obtain

$$\hat{\iota} = \inf\{\widetilde{\hat{\mathcal{J}}}_y - \delta(y)\hat{\mathcal{J}}_y : y \in V\} \geq \frac{\iota}{2} > 0,$$

since $\hat{\mathcal{J}}$ is an extension of $\mathcal{J}$ on Λ, and the function δ is defined by $\hat{J}$ and $D(x)$ according to Remark 2.11. Finally, by taking a sufficiently small $\varepsilon > 0$ in the choice of the C^1 approximation $\bar{J}$, we also get

$$\bar{\iota} = \inf\{\widetilde{\bar{\mathcal{J}}}_y - \delta(y)\bar{\mathcal{J}}_y : y \in V\} \geq \frac{\hat{\iota}}{2} \geq \frac{\iota}{4} > 0.$$

From Theorem 2.7 and Proposition 2.21, we know that this is a necessary and sufficient condition for strict $\bar{\mathcal{J}}$-separation of $A_t(x)$ over V.

2.5.4. *Strict separation for nearby flows/cocycles.* Given a vector field X on M and a linear multiplicative cocycle $A_t(x)$ on a vector bundle E over M, for a C^1 close vector field Y and a C^0 close cocycle $B_t(x)$ over Y, the infinitesimal generator $D_{B,Y}(x)$ of $B_t(x)$ will be a linear map close to the infinitesimal generator $D(x)$ of $A_t(x)$ at x. That is, given $\varepsilon > 0$ we can find a C^1 neighborhood $\mathcal{V}$ of X and a C^0 neighborhood $\mathcal{W}$ of the cocycle A such that

$$(Y, B) \in \mathcal{V} \times \mathcal{W} \implies \|D_{B,Y}(x) - D(x)\| < \varepsilon, \quad x \in M.$$

Hence, since δ also depends continuously on the infinitesimal generator, we obtain

$$\tilde{\iota} = \inf\{\tilde{\bar{\mathfrak{J}}}_y - \delta_{B,Y}(y)\bar{\mathfrak{J}}_y : y \in V, Y \in \mathcal{V}, B \in \mathcal{W}\} \geq \frac{\bar{\iota}}{2} > 0.$$

This shows that we have strict $\bar{\mathfrak{J}}$ separation for all nearby cocycles over all C^1-close enough vector fields over the same neighborhood V of the original invariant attracting set Λ.

This completes the proof of Theorem 2.16.

2.6. **Characterization of the splitting through the function δ.** We now use the area under the function δ to characterize different dominated splittings that may arise in our setting.

Theorem 2.22. *Let Λ be a compact invariant set for X_t with a strictly $\mathfrak{J}$-separated linear cocycle $A_t(x)$ over Λ. Let $E_\Lambda = F_- \oplus F_+$ be the dominated splitting for $A_t(x)$, at each $x \in \Lambda$. Then we have:*

(1) *$F_- \oplus F_+$ is hyperbolic if, and only if, for all $x \in \Lambda$*

$$\lim_{t\to+\infty} \Delta(x,t) = +\infty \quad \textit{and} \quad \lim_{t\to-\infty} \Delta(x,t) = -\infty;$$

this is, F_- is uniformly contracted and F_+ is uniformly expanded.

(2) *$F_- \oplus F_+$ is partially hyperbolic with F_- uniformly contracting and F_+ not uniformly expanding if, and only if*

$$\lim_{t\to+\infty} \Delta(x,t) = -\infty, \ \textit{for all } x \in \Lambda \quad \textit{and}$$
$$\Delta(x,t) \ \textit{is bounded below for } t \to -\infty \ \textit{for some } x \in \Lambda.$$

(3) *$F_- \oplus F_+$ is partially hyperbolic with F_- not uniformly contracting and F_+ uniformly expanding if, and only if*

$$\lim_{t\to-\infty} \Delta(x,t) = -\infty, \ \textit{for all } x \in \Lambda \quad \textit{and}$$
$$\Delta(x,t) \ \textit{is bounded below for } t \to +\infty \ \textit{for some } x \in \Lambda.$$

In the proof we use the following useful equivalence.

Lemma 2.23. *Let $F \subset E$ be a continuous $A_t(x)$-invariant subbundle of the finite dimensional vector bundle E with compact base Λ. Then, there are constants $K, \omega > 0$ satisfying for $\vec{0} \neq v \in F_x, x \in \Lambda, t > 0$*

$$\|A_t(x)v\| \leq Ke^{-\omega t}\|v\| \quad (\|A_{-t}(x)v\| \leq Ke^{-\omega t}\|v\|, \ \textit{respectively})$$

if, and only if, for every $x \in \Lambda$ and $\vec{0} \neq v \in E_x$

$$\lim_{t\to+\infty} \|A_t(x)v\| = 0 \quad (\lim_{t\to+\infty} \|A_{-t}(x)v\| = 0, \ \textit{respectively}).$$

Proof. See e.g. [22]. ∎

We also need to obtain a relation between $\mathcal{J}$ and the norm over the invariant subspaces. For a strictly $\mathcal{J}$-separated flow we have for every $t > 0$

$$\min_{v \in C_0(x) \cup C_+(x)} \frac{\mathcal{J}(A_t(x)v)}{\|A_t(x)v\|^2} = \inf_{v \in C_+(x)} \frac{\langle JA_t(x)v, A_t(x)v\rangle}{\langle A_t(x)v, A_t(x)v\rangle} > 0$$

since $\mathbb{S}^n_x \cap (C_0(x) \cup C_+(x)) = \{v \in C_0(x) \cup C_+(x) : \|v\| = 1\}$ is a compact subset of the unit sphere $\mathbb{S}^n_x$ at E_x and its image by $A_t(x)$ is a compact subset in the interior of $C_+(X_t(x))$.

Then the following

$$\upsilon(x,t) := \inf_{v \in F_+(x)} \frac{\mathcal{J}(A_t(x)v)}{\|A_t(x)v\|^2}$$

is a positive function $\upsilon : U \times \mathbb{R} \to (0, +\infty)$. In a similar way we define the following useful positive functions on $U \times \mathbb{R}$

$$\xi(x,t) := \sup_{v \in F_+(x)} \frac{\mathcal{J}(A_t(x)v)}{\|A_t(x)v\|^2}, \quad \zeta(x,t) := \inf_{w \in F_-(x)} \frac{|\mathcal{J}(A_t(x)w)|}{\|A_t(x)w\|^2}$$

$$\text{and} \quad \varrho(x,t) := \sup_{w \in F_-(x)} \frac{|\mathcal{J}(A_t(x)w)|}{\|A_t(x)w\|^2}.$$

Hence each $\upsilon(x,t), \xi(x,t), \zeta(x,t)$ and $\rho(x,t)$ are positive continuous functions for all $x \in \Lambda, t \in \mathbb{R}$. The compactness of Λ now implies that $\xi(x,t) \leq \bar{\xi} < \infty$, $\rho(x,t) \leq \bar{\rho} < \infty$, $\upsilon(x,t) \geq \underline{\upsilon} > 0$ and $\zeta(x,t) \geq \underline{\zeta} > 0$ for all $t \in \mathbb{R}, x \in \Lambda$. Then by standard linear algebra we obtain the following.

Lemma 2.24. *There exists a constant $C > 0$ such that, for each $x \in \Lambda$ satisfying $\mathcal{J}(X(x)) \geq 0$, for every pair of non-zero vectors $(w,v) \in F_-(x) \times F_+(x)$ and for every $t \in \mathbb{R}$*

$$\underline{\zeta}\|A_t(x)w\|^2 \leq |\mathcal{J}(A_t(x)w)| \leq \bar{\rho}\|A_t(x)w\|^2,$$
$$\underline{\upsilon}\|A_t(x)v\|^2 \leq \mathcal{J}(A_t(x)v) \leq \bar{\xi}\|A_t(x)v\|^2 \quad \textit{and}$$
$$\frac{1}{C}\sqrt{\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)}} \leq \frac{\|A_t(x)w\|}{\|A_t(x)v\|} \leq C\sqrt{\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)}}.$$

Now we are ready to prove Theorem 2.22.

Proof of Theorem 2.22. First we assume that $\lim_{t\to\pm\infty} \Delta(x,t) = \pm\infty, x \in \Lambda$ and note that from Theorem 2.7(4)

$$\lim_{t\to+\infty} \mathcal{J}(A_t(x)v_+) = +\infty, \quad v_+ \in F_+(x), x \in \Lambda$$

and from Theorem 2.7(2) for $v_- \in F_-(x), x \in \Lambda$

$$\lim_{t\to+\infty} \frac{|\mathcal{J}(v_-)|}{|\mathcal{J}(A_{-t}(x)v_-)|} \leq \lim_{t\to+\infty} \exp \Delta(x,-t) = 0, \quad \text{thus} \quad \lim_{t\to+\infty} |\mathcal{J}(A_{-t}(x)v_-)| = +\infty.$$

Hence from the comparison results given in Lemma 2.24 we see that vectors in F_+ have norm which grows without limit as t goes to $+\infty$ and vector in F_- are also expanded without

bound for negative time. This is equivalent to hyperbolicity for a finite dimensional vector bundle over a compact base, from Lemma 2.23.

Reciprocally, let us assume that F_+ is formed by uniformly expanded vectors for positive time, and F_- by uniformly expanded vectors for negative time. From Theorem 2.16 we can find a family $\bar{\mathcal{J}}$ of quadratic forms on a neighborhood V admitting a constant $\omega > 0$ and a function $\kappa : \Lambda \to (0, 2)$ such that for all $x \in \Lambda$ and $t > 0$

$$\bar{\mathcal{J}}(A_t(x)v_+) \geq \kappa_{x,t} e^{\omega t} \bar{\mathcal{J}}(v_+) \quad \text{and} \quad |\bar{\mathcal{J}}(A_{-t}(x)v_-)| \geq \kappa_{x,-t} e^{\omega t} |\bar{\mathcal{J}}(v_-)|, \quad v_\pm \in F_\pm(x). \tag{2.15}$$

Hence for $t > 0$

$$\frac{1}{t}\left[\log \bar{\mathcal{J}}(A_t(x)v_+) - \log \bar{\mathcal{J}}(v_+)\right] \geq \omega + \frac{1}{t}\left[\log \kappa_{x,t} - \log \kappa_{x,0}\right]$$

and letting $t \to 0^+$ we obtain $\partial_t \log \bar{\mathcal{J}}(A_t(x)v_+) \mid_{t=0} \geq \omega + \partial_t \log \kappa_{x,t} \mid_{t=0}$ or

$$\frac{\partial_t \bar{\mathcal{J}}(A_t(x)v_+) \mid_{t=0}}{\bar{\mathcal{J}}(v_+)} \geq \omega + \frac{\partial_t \kappa_{x,t} \mid_{t=0}}{\kappa_{x,0}} \geq \frac{\omega}{2}.$$

Therefore, $\partial_t \bar{\mathcal{J}}(A_t(x)v_+) \mid_{t=0} \geq \frac{\omega}{2} \bar{\mathcal{J}}(v_+)$. Analogously, we obtain the same result with a vector $v_- \in F_-(x)$ in the place of v_+. Therefore we may take $\delta \equiv \omega/4 > 0$ on Λ and then $\lim_{t\to\pm\infty} \Delta(x, t) = \pm\infty$ as we want, due to the equivalence between $\mathcal{J}$ and $\bar{\mathcal{J}}$ by Remark 2.18.

For item (2), we first assume that $\lim_{t\to+\infty} \Delta(x, t) = -\infty$ for all $x \in \Lambda$. Hence for $\vec{0} \neq v_- \in F_-(x), x \in \Lambda$

$$\lim_{t\to+\infty} |\mathcal{J}(A_t(x)v_-)| \leq \lim_{t\to+\infty} |\mathcal{J}(v_-)| \exp \Delta(x, t) = 0,$$

and we see that vectors in F_- are uniformly contracted by the action of the cocycle, following Lemma 2.23. Moreover, if there exists $x_0 \in \Lambda$ such that $\{\Delta(x_0, t) : t < 0\}$ is bounded below, then F_+ cannot be a uniformly expanded subbundle. Indeed, otherwise $F_- \oplus F_+$ would be a hyperbolic splitting and, from item (1) already proved, it would follow that $\lim_{t\to-\infty} \Delta(x_0, t) = -\infty$, contradicting the boundedness assumption on x_0.

Reciprocally, if $F_- \oplus F_+$ is a partially hyperbolic splitting with uniformly contracting direction along F_-, then from Theorem 2.16 we can find a family $\bar{\mathcal{J}}$ of quadratic forms in a neighborhood V of Λ such that $A_t(x)$ is strictly $\bar{\mathcal{J}}$-separated and satisfies the relations in items (1-3) of Theorem 2.16.

Arguing as in the proof of the first item above, from item (1) of Theorem 2.16 we obtain $\partial_t \mathcal{J}(A_t(x)v_-) \mid_{t=0} \leq -\omega/2$ so we can take $\delta \equiv -\omega/4 < 0$ on Λ. Then $\lim_{t\to+\infty} \Delta(x, t) = -\infty$ and $\Delta(x, t)$ is bounded below when $t \to -\infty$, as stated, again using Remark 2.18.

Finally, we note that if $F_- \oplus F_+$ is a partially hyperbolic splitting as stated in item (3), then $F_+ \oplus F_-$ is a partially hyperbolic splitting as considered in item (2) for the inverse cocycle $A(x, -t)$. As explained in Remark 2.10, in this case $A(x, -t)$ is strictly $(-\mathcal{J})$-separated. Hence we have the function δ playing the same role as before, since we have to exchange the role of the positive and negative cones, but we also change the time direction. Thus we can reverse time in all the asymptotic relations for $\Delta(x, t)$ from item (2). $\square$

3. Partial hyperbolicity - Proof of Theorem A

Now we prove Theorem A. We show that strict $\mathcal{J}$-separation of a $\mathcal{J}$-non-negative flow X_t on a trapping region U implies the existence of a dominated splitting and that the dominating bundle (the one with the weakest contraction) is necessarily uniformly contracting. That is, we have in fact a partially hyperbolic splitting.

The strategy is to consider the derivative cocycle DX_t of the smooth flow X_t in the place of $A_t(x)$ and use the results of Section 2.2. In this setting we have that the infinitesimal generator is given by $D(x) = DX(x)$ the spatial derivative of the vector field X. Since the direction of the flow $E^X_x := \mathbb{R} \cdot X(x) = \{s \cdot X(x) : s \in \mathbb{R}\}$ is DX_t-invariant for all $t \in \mathbb{R}$, if U is a trapping region where X_t is $\mathcal{J}$-separated and $\mathcal{J}(X(x)) \geq 0$ for some $x \in U$, then $\mathcal{J}(DX_t(X(x))) \geq 0$ for all $t > 0$ and this function is bounded. This implies, from (2.10), that $\Delta(x,t) = \int_0^t \delta \circ X^s(x)\, ds$ must be a bounded function for $t > 0$.

Remark 3.1. In the same setting, if $\mathcal{J}(X(x)) \leq 0$, then $\Delta(x,t)$ is bounded for $t < 0$. This restricts the possible functions $\delta(x)$ for $\mathcal{J}$-separated flows.

We start with the following useful lemma, whose proof follows [32, Chapter 10].

Let U be a trapping region for the flow of X and $\Lambda(U) = \Lambda_X(U)$ the corresponding attracting set.

Lemma 3.2. *There exists a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}^1(M)$ and an open neighborhood V of $\Lambda(U)$ such that V is a trapping region for all $Y \in \mathfrak{U}$, that is, there exists $t_0 > 0$ for which*

- $Y_t(V) \subset V \subset U$ *for all* $t > 0$*;*
- $\overline{Y_t(V)} \subset V$ *for all* $t > t_0$*; and*
- $\overline{Y_t(V)} \subset U$ *for all* $t > 0$.

Corollary 3.3. *Let X_t be strictly $\mathcal{J}$-separated on U. Then there exist a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}^1(M)$ and a neighborhood V of $\Lambda(U)$ such that V is a trapping region for every $Y \in \mathfrak{U}$ and each $Y \in \mathfrak{U}$ is strictly $\mathcal{J}$-separated in V.*

Proof. The assumption implies that $\tilde{\mathcal{J}}_x = \tilde{\mathcal{J}}^X_x > \alpha(x)\mathcal{J}$ for all $x \in U$. Let $\mathfrak{U}$ and V be the neighborhoods of X and Λ given by Lemma 3.2. Let also $Y \in \mathfrak{U}$ be fixed.

Writing $\tilde{J}^Y_x := J \cdot DY(x) + DY(x)^* \cdot J$, we can make the norm $\|\tilde{\mathcal{J}}_x - \tilde{\mathcal{J}}^Y_x\|$ smaller than

$$\frac{1}{2}\min\left\{\inf_{v\in C_+(x)} \frac{\langle \tilde{J}_x v, v\rangle}{\langle Jv, v\rangle} - \alpha(x) : x \in \overline{U_0}\right\}$$

for all $x \in V$ by shrinking $\mathfrak{U}$ if needed. This ensures that there exists $\beta : \overline{V} \to \mathbb{R}$ such that $\tilde{\mathcal{J}}^Y_x > \beta(x)\mathcal{J}$ for all $x \in \overline{V}$, so Y is strictly $\mathcal{J}$-separated on V. $\square$

From Section 2.4 we know that there exists a continuous dominated splitting $F^Y_-(x) \oplus F^Y_+(x)$ of T_xM for $x \in \Lambda_Y(U)$, with respect to Y^t for all $Y \in \mathfrak{U}$.

The strict $\mathcal{J}$-separation on U for X_t implies that any invariant subbundle of T_xM along an orbit of the flow X_t must be contained in $F_\pm(x)$. In particular, the characteristic space corresponding to the flow direction is contained in $F_+(x)$.

The following lemma, however simple, is crucial for our arguments. We recall that a non-trivial periodic orbit γ of X, is a orbit for which there exists $\tau > 0$ minimal such that $X_\tau(p) = p$ and $X_t(p) \neq p$, for every point $p \in \gamma$ and for all $0 < t < \tau$, and we say that p is a *τ-periodic point.*

Lemma 3.4. *Let $p \in U$ be a non-trivial τ-periodic point of $Y \in \mathfrak{U}$. If Y is $\mathcal{J}$-non-negative (non-positive), then $\Delta(p, \tau) < 0$ ($\Delta(p, \tau) > 0$, respectively).*

Proof. Let us assume that Y is a non-negative vector field. We use that the orbit of p is a closed smooth curve together with item (3) of Theorem 2.7 to obtain,

$$1 = \frac{\mathcal{J}(Y(Y_\tau p))}{\mathcal{J}(Y(p))} > \exp \Delta(p, \tau).$$

Thus, $\Delta(p, \tau) < 0$, as stated. If Y is non-positive, the last inequality is reversed and so we get $\Delta(p, \tau) > 0$ in this case, as in the statement of the lemma. $\square$

We can apply a similar argument to the case of a "degenerate periodic orbit", i.e., an equilibrium point of X in U.

Lemma 3.5. *Let $\sigma \in \mathrm{Sing}(X|_U)$ be such that $W^s(\sigma) \neq \{\sigma\}$, for a non-negative strictly $\mathcal{J}$-separated flow $X \in \mathfrak{X}^1(M)$ on U. Then we can choose $\delta(\sigma) < 0$ so that $\tilde{\mathcal{J}}_\sigma > \delta(\sigma)\mathcal{J}$.*

Proof. Let us take $x \in W^s_{loc}(\sigma) \setminus \{\sigma\}$ and note that for all $s > 0$ and big enough $\tau > s$

$$1 > \frac{\mathcal{J}(Y(Y_\tau(x)))}{\mathcal{J}(Y(Y_s(x)))} > \exp(\Delta(x, \tau) - \Delta(x, s)) = \exp \int_s^\tau \delta(Y_t(x))\, dt, \tag{3.1}$$

since $\mathcal{J}(Y(Y_t(x))) > 0$ for all $t > 0$ and $\lim_{t\to+\infty} \mathcal{J}(Y(Y_t(x))) = 0$.

We assume that δ is chosen as in Remark 2.11, so that it is a continuous function. This shows that there exists a sequence $\tau_n \to +\infty$ such that $\delta(X_{\tau_n}(x)) < 0$ for all $n \in \mathbb{Z}^+$ and, since $\lim_{t\to+\infty} X_{\tau_n}(x) = \sigma$ we get $\delta(\sigma) \leq 0$.

However, we cannot have $\delta(\sigma) = 0$. For otherwise, since we have strict $\mathcal{J}$-monotonicity, by Remark 2.11 we would be able to choose $\delta(\sigma) > 0$ and so $\delta(X_\tau(x)) > 0$ for all $\tau > s$ for some big enough $s > 0$, contradicting (3.1). $\square$

As a consequence we deduce the following important intermediate step towards proving the existence of a partially hyperbolic splitting on $\Lambda(U)$.

Corollary 3.6. *For every $Y \in \mathfrak{U}$ with Y non-negative, and every periodic point $p \in U$ (or singularity $p = \sigma \in U$ for Y such that $W^s(\sigma) \neq \{\sigma\}$) we have*

1. $\lim_{t\to+\infty} \|DY_t w\| = 0$ *for all* $w \in F_-(p)$*;*
2. $\lim_{t\to+\infty} \frac{\|DY_t w\|}{\|DY_t v\|} = 0$ *for each non-zero pair* $(w, v) \in F_-(p) \times F_+(p)$*.*

Proof. If $Y(p) \in C_+(p)$ then, for $(w, v) \in F_-(p) \times F_+(p)$, since by Lemma 3.4 we have $\Delta(p, \tau) < 0$, we deduce that $\lim_{t\to+\infty} \Delta(p, t) = -\infty$. Hence by Lemma 2.24 and item (4b) of Proposition 2.7 we obtain the first statement.

For the second statement we use again Lemma 2.24 and item (5) of Proposition 2.7. For the equilibrium case, just use Lemma 3.5 together with Lemma 2.24. □

This shows that every periodic orbit on U, for all vector fields Y which are C^1 close to X, admits a partially hyperbolic splitting, if the vector fields are non-negative, and the index of the periodic orbit is at least the dimension of the negative space. Indeed, it is enough to recall that the invariant subspaces of $DY_\tau(p) : T_pM \to T_pM$, if p is a periodic orbit of Y in U and $\tau > 0$ its minimal period, are contained either in $F_-(p)$ or in $F_+(p)$. In addition, using Lemma 3.5, we also arrive at the same conclusion for every equilibrium $\sigma \in U$ whose stable manifold is non-trivial: the dimension of the stable manifold of σ is at least the dimension of the negative sub-bundle.

To obtain a partially hyperbolic splitting we need that the contracting rates along the strong-stable directions at periodic orbits be uniform on a C^1 neighborhood of X.

3.1. **Strictly $\mathcal{J}$-separation implies partial hyperbolicity.** Now we argue that, in this setting, the flow on $\Lambda(U)$ has a partially hyperbolic splitting. For that, for every periodic point $p \in U$ for a C^1 nearby flow Y_t, we consider the subspace $E^{ss}(p) := E^s(p) \cap F_-(p)$ of negative contracting eigenvectors of $DY_\tau(p)$, where $\tau > 0$ is the minimal period of p. We claim that the contracting rate is uniform on the period, as follows.

Proposition 3.7. *There are a neighborhood $\mathcal{V} \subset \mathcal{U}$ of X in the C^1 topology, and constants $0 < \lambda < 1$ and $c > 0$, such that, for every $Y \in \mathcal{V}$, if p is a periodic point of Y_t in U_0 and τ_p is the minimal period of p, then $\|DY_{\tau_p} \mid E^{ss}(p)\| < \lambda^{\tau_p}$.*

Proof. The proof is by contradiction using the perturbation lemma of Franks. We suppose, by contradiction, that given $\delta > 0$ small, there is $Y \in \mathfrak{X}^\infty(M)$ arbitrarily C^1 close to X, and a periodic orbit p of Y with period $\tau = \tau_p > 0$, such that $\|DY_\tau \mid E^{ss}(p)\| \geq (1-\delta)^\tau$.

Let A_t be the one-parameter family of linear maps

$$A_t = (1-2\delta)^{-t} \cdot DY_t(p), \quad 0 \leq t \leq \tau.$$

By construction A_t preserves the flow direction and the eigenspaces of DY_t. Moreover

$$\|\partial_h A_{t+h} A_t^{-1}|_{h=0} - DY(Y_t(p))\| < -\log(1-\delta)\,.$$

Since we can take δ as close to 0 as needed, the inequality above together with $Y \in C^\infty$ imply that A_t satisfies Frank's Lemma; see e.g. [4, 7] for its statement, proof and applications.

Hence there exists $Z \in \mathfrak{X}^1(M)$, which is C^1 near Y and so we can assume that $Z \in \mathcal{U}$, such that p is a τ-periodic point of Z, and $DZ_t(Z_t(p)) = A_t$, for $0 \leq t \leq \tau$. By definition of A_t we get $\|DZ_\tau | E^{ss}(p)\| > 1$, implying that p is a periodic orbit for Z with some expanding direction along $E^{ss}(p)$. This contradicts Corollary 3.6 and completes the proof. □

Now we can apply the Ergodic Closing Lemma to conclude that on the non-wandering subset $\Lambda_X(U) \cap \Omega(X)$ of $\Lambda_X(U)$ the dominated splitting obtained before is, in fact, a partially hyperbolic splitting with a uniformly contracting and dominated subbundle; again this follows the same arguments of [27], see also [4] for an extended presentation. We present the main result in the following subsection.

Remark 3.8. From Lemma 2.24 and Proposition 2.7 we see that a sufficient condition to obtain partial hyperbolicity is $\lim_{t\to+\infty}\Delta(x,t)=-\infty$ for all $x\in\Lambda(U)$. To obtain partial hyperbolicity for the non-wandering part of $\Lambda(U)$, it is enough to get $\Delta(p,\tau)<0$, for every periodic point $p\in U$ of X_t with period $\tau>0$. Finally, a sufficient condition for all the above is $\delta(x)<-a<0$, for each $x\in\Lambda(U)$.

We observe that if $\lim_{t\to+\infty}\Delta(x,t)=-\infty$ for $x\in\Lambda(U)$, then the flow direction must be non-negative.

Let Λ be a compact invariant set for a flow X of a C^1 vector field X on M.

Lemma 3.9. *Given a continuous splitting $T_\Lambda M=E\oplus F$ such that E is uniformly contracted, then $X(x)\in F_x$ for all $x\in\Lambda$.*

Proof. We denote by $\pi(E_x):T_xM\to E_x$ the projection on E_x parallel to F_x at T_xM, and likewise $\pi(F_x):T_xM\to F_x$ is the projection on F_x parallel to E_x. We note that for $x\in\Lambda$

$$X(x)=\pi(E_x)\cdot X(x)+\pi(F_x)\cdot X(x)$$

and for $t\in\mathfrak{R}$, by linearity of DX_t and DX_t-invariance of the splitting $E\oplus F$

$$\begin{aligned}DX_t\cdot X(x)&=DX_t\cdot\pi(E_x)\cdot X(x)+DX_t\cdot\pi(F_x)\cdot X(x)\\&=\pi(E_{X_t(x)})\cdot DX_t\cdot X(x)+\pi(F_{X_t(x)})\cdot DX_t\cdot X(x)\end{aligned}$$

Let z be a limit point of the negative orbit of x, that is, we assume that there is a strictly increasing sequence $t_n\to+\infty$ such that $\lim_{n\to+\infty}x_n:=\lim_{n\to+\infty}X_{-t_n}(x)=z$. Then $z\in\Lambda$ and, *if* $\pi(E_x)\cdot X(x)\neq\vec{0}$ we get

$$\begin{gathered}\lim_{n\to+\infty}DX_{-t_n}\cdot X(x)=\lim_{n\to+\infty}X(x_n)=X(z)\quad\text{but also}\\ \|DX_{-t_n}\cdot\pi(E_x)\cdot X(x)\|\ge ce^{\lambda t_n}\|\pi(E_x)\cdot X(x)\|\xrightarrow[n\to+\infty]{}+\infty.\end{gathered}\tag{3.2}$$

This is possible only if the angle between E_{x_n} and F_{x_n} tends to zero when $n\to+\infty$.

Indeed, using the Riemannian metric on T_yM, the angle $\alpha(y)=\alpha(E_y,F_y)$ between E_y and F_y is related to the norm of $\pi(E_y)$ as follows: $\|\pi(E_y)\|=1/\sin(\alpha(y))$. Therefore

$$\begin{aligned}\|DX_{-t_n}\cdot\pi(E_x)\cdot X(x)\|&=\|\pi(E_{x_n})\cdot DX_{-t_n}\cdot X(x)\|\\&\le\frac{1}{\sin(\alpha(x_n))}\cdot\|DX_{-t_n}\cdot X(x)\|\\&=\frac{1}{\sin(\alpha(x_n))}\cdot\|X(x_n)\|\end{aligned}$$

for all $n\ge1$. Hence, if the sequence (3.2) is unbounded, then $\lim_{n\to+\infty}\alpha(X_{-t_n}(x))=0$.

However, since the splitting $E\oplus F$ is continuous over the compact Λ, the angle $\alpha(y)$ is a continuous and positive function of $y\in\Lambda$, and thus must have a positive minimum in Λ. This contradiction shows that $\pi(E_x)\cdot X(x)$ is always the zero vector and so $X(x)\in F_x$ for all $x\in\Lambda$. $\square$

3.1.1. *Subadditive functions of the orbits of a flow and exponential growth.* Let U be an open subset of M such that $\overline{X_t(U)} \subset U$ for all $t > 0$ and $\Lambda_Y(U)$ the corresponding attracting set for $Y \in \mathcal{V}$, where $\mathcal{V}$ is a C^1 neighborhood of $X \in \mathfrak{X}^1(M)$. We say that a family of functions $\phi_Y : \mathbb{R} \times \Lambda_Y(U) \to \mathbb{R}$ is *subadditive* if

$$\phi_Y(t+s,x) \le \phi_Y(s, X_t(x)) + \phi_Y(t,x), \quad \text{for all} \quad t,s \in \mathbb{R}, x \in \Lambda \quad \text{and} \quad Y \in \mathcal{V}.$$

Let ϕ_Y be a family of continuous subadditive functions on the subsets Λ_Y for $Y \in \mathcal{V}$ such that for all $x \in \Lambda_Y$ and every $Y \in \mathcal{V}$

- $\phi_Y(0,x) = 0$;
- $D_Y(x) = \limsup_{h\to 0}(\phi_Y(h,x)/h) < \infty$ depends continuously on x;
- $\Lambda_X(U) \cap S(X)$ is discrete;
- $D(\sigma) < 0$ for each $\sigma \in \Lambda_X(U) \cap S(X)$.

Moreover $\phi_Y(s,y)$ depends continuously on (s,y,Y) as follows: if $Y_n \xrightarrow[n\to\infty]{C^1} X$, $y_n \in \Lambda_{Y_n}(U)$, $s_n \in \mathbb{R}$ are such that $y_n \xrightarrow[n\to\infty]{} x \in \Lambda_X(U)$ and $s_n \xrightarrow[n\to\infty]{} t \in \mathbb{R}$, then

$$\phi_{Y_n}(s_n, y_n) \xrightarrow[n\to\infty]{} \phi_X(t,x).$$

Theorem 3.10. *Let us assume that*

(1) *there exists $T_0 > 0$ and $a > 0$ such that, for each $Y \in \mathcal{V}$ and for every t_p-periodic point $p \in \Lambda_Y(U)$ for Y such that $t_p \ge T_0$, we have $\phi_Y(t_p,p) \le -at_p$; and*
(2) *$\phi_Y(t_p,p) < 0$ for all $p \in \mathrm{Per}(Y) \cap \Lambda_X(U)$.*

Then for every compact invariant subset Γ of $\Lambda_X(U)$

$$\phi_X(t,x) \le -at \text{ for all } t > 0, \text{ every } x \in \Gamma \text{ and all } X \in \mathcal{V}.$$

Proof. See e.g. [4, Section 4.3.1]. □

We apply this result to the subadditive family

$$\phi_Y(t,x) := \log \|DY_t|_{F_-^Y(x)}\|, \quad Y \in \mathcal{V}, x \in \Lambda_Y(U), \ t \in \mathbb{R}.$$

It is straightforward to check the conditions of Theorem 3.10 by the smoothness of the vector field, Proposition 3.7 and the fact that we assume that each equilibrium inside U is isolated and has a non-trivial stable manifold with dimension bigger or equal to the dimension of $F_-(\sigma)$. We conclude that the dominating splitting $F_- \oplus F_+$ over $\Gamma = \Lambda_Y(U)$ is such that the F_- is uniformly contracted by the flow of every vector field Y in a C^1 neighborhood of X.

3.1.2. *Extending the conclusion to the maximal invariant subset.* We can extend the conclusion to the whole $\Lambda_X(U)$ because the limit set is always contained in the non-wandering set $\Omega(X)$.

Lemma 3.11. *If a compact invariant set Λ for a flow X_t is such that $\Lambda \cap \Omega(X)$ is partially hyperbolic, then Λ is also a partially hyperbolic set.*

Proof. Since the non-wandering set contains all periodic points and singularities, and also all limit points, that is, for all $x \in \Lambda$ we have $\omega_X(x) \cup \alpha_X(x) \subset \Omega(X)$ and $S(X) \cup \mathrm{Per}(X) \subset \Omega(X)$, then for each $\varepsilon > 0$ and $x \in \Lambda$ there exists $t_x > 0$ such that

$$|t| > t_x \implies X_t(x) \in B_\varepsilon(\Lambda \cap \Omega(X))$$

where $B_\varepsilon(A) := \cup_{x \in A} B_\varepsilon(x)$ for any subset A is the ε-neighborhood of A. The compactness of Λ and continuity of the flow X_t ensure that there exists $T > 0$ such that $X_T(\Lambda) \cup X_{-T}(\Lambda) \subset B_\varepsilon(\Lambda \cap \Omega(X))$.

Now, since $K := \Lambda \cap \Omega(X)$ is a partially hyperbolic subset for X_t, we can find two fields of cones with small width C^s, C^{cu} over K which are strictly Df-invariant and the length of vectors in C^s is uniformly contracted. We can extend the field of cones to a neighborhood $B_\varepsilon(K)$ of K, denoted by the same letters, such that there exists $C, \lambda > 0$ satisfying for $s > 0$

$$y \in \bigcap_{0 \le t \le s} X_{-t}(B_\varepsilon(K)) \implies \overline{DX_t C^{cu}(y)} \subset C^{cu}(X_t(y)), \quad \forall 0 < t \le s;$$

$$y \in \bigcap_{0 \le t \le s} X_t(B_\varepsilon(K)) \implies \overline{DX_t C^s(y)} \subset C^s(X_t(y)), \quad \forall -s \le t < 0$$

$$\text{and} \quad \|DX_t(v)\| \le Ce^{-\lambda t}\|v\|, \forall -s \le t < 0, v \in C^s(y).$$

We can now define cones on $z \in \Lambda \setminus B_\varepsilon(K)$ by

$$C^s(z) := DX_{-T}\big(C^s(X_T(z))\big) \quad \text{and} \quad C^{cu}(z) := DX_T\big(C^{cu}(X_{-T}(z))\big).$$

It is easy to see that these cones are DX_t invariant for all $t > T$ and that the vectors on the C^s cone are contracted uniformly. This shows that Λ is a partially hyperbolic set. $\square$

At this point, we have concluded the proof of sufficiency in the statement Theorem A.

The necessary part of the statement of Theorem A is a simple consequence of Theorem 2.16 applied to the cocycle DX_t acting on the vector bundle $T_U M$, the tangent bundle on the trapping region U.

This completes the proof of Theorem A.

3.2. **Uniform Hyperbolicity.** By using Theorem A, we are able to present the proof of the Corollaries B and C.

Proof of Corollary B. Let $X \in \mathfrak{X}^1(M)$ and Λ be the maximal invariant set of a trapping region U for X. Consider $\mathcal{J}, \mathcal{G}$ two differentiable fields of non-degenerated quadratic forms on U with constant indices s and $n - s - 1$, respectively, where $n = \dim M$ and $s < n - 2$. Since $\Lambda \cap \mathrm{Sing}(X) = \emptyset$, the flow direction $X(x)$ is non-zero for all point $x \in \Lambda$ and generates an invariant line bundle E^X over Λ.

On the one hand, by Remark 2.10, $-X$ is a non-negative strictly separated with respect to $-\mathcal{G}$ on Λ. Then Theorem A implies that there is a partially hyperbolic splitting $T_\Lambda M = E^{cs} \oplus E^u$ with the subbundle E^u uniformly expanding, and dimensions $\dim E^{cs} = s + 1$ and $\dim E^u = n - s - 1$. Thus, by Lemma 3.9, the flow direction E^X is contained in E^{cs}.

On the other hand, with analogous arguments, we prove that, for $\mathcal{J}$, Λ has a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^{cu}$, with E^s uniformly contracting, and so $E^X \subset E^{cu}$, with dimensions $\dim E^s = s$ and $\dim E^{cu} = n - s$.

Moreover we clearly have $E^s \cap E^{cu} = \{0\}$ and $E^u \cap E^{cs} = \{0\}$. Hence we have the following dominated splittings

$$(E^s \oplus E^X) \oplus E^u = E^s \oplus (E^X \oplus E^u) = T_\Lambda M,$$

with $\dim E^{cs} = \dim(E^s \oplus E^X)$ and $\dim E^{cu} = \dim(E^s \oplus E^X)$. By uniqueness of dominated splittings with the same dimensions we obtain $E^{cu} = E^X \oplus E^u$ and $E^{cs} = E^s \oplus E^X$, and the splitting of $T_\Lambda M$ above is a hyperbolic splitting. □

Proof of Corollary C. Consider M a closed Riemannian manifold with dimension $n \geq 3$ and $X \in \mathfrak{X}^1(M)$ a incompressible vector field.

Let $\mathcal{J}$ be a field of non-degenerate quadratic forms on M, with constant index $\operatorname{ind}(\mathcal{J}) = \dim(M) - 2$, such that X_t is a non-negative $\mathcal{J}$-separated flow.

By Theorem A and the hyphotesis on $\operatorname{ind}(\mathcal{J})$, we obtain a partially hyperbolic splitting $TM = E \oplus F$, with $\dim E = \dim(M) - 2$ and $\dim F = 2$ and E uniformly contracted. Hence, as the flow is volume-preserving, the area along the two-dimensional direction F is expanded. Indeed, the angle $\theta(E_x, F_x)$ between the subbundles is uniformly bounded away from zero (by domination of the splitting; see [28]) and so

$$1 = |\det DX_t(x)| = |\det DX_t\mid_{E_x}| \cdot |\det DX_t\mid_{F_x}| \cdot \sin\theta(E_{X_t(x)}, F_{X_t(x)})$$

which for $t < 0$ ensures that

$$|\det DX_t\mid_{F_x}| \leq \sin\theta_0 \cdot |\det DX_t\mid_{E_x}|^{-1} \xrightarrow[t\to-\infty]{} 0.$$

Thus, M is a singular-hyperbolic set for X. Moreover, there can be no singularities, since they cannot be in the interior of a singular-hyperbolic set; see Doering [12] and Morales-Pacifico-Pujals [26] in dimension three; Bautista-Morales [5] and Vivier [42] for higher dimensions; or [4, Chapter 4]. Hence M is a singular-hyperbolic set for X_t without singularities. Therefore X is an Anosov flow. □

4. Sectional-hyperbolicity - proof of Theorem D

Here we prove Theorem D. We assume that X is a C^1 vector field in an open trapping region U with a smooth family $\mathcal{J}$ of non-degenerate quadratic forms such that X is non-negative and strictly $\mathcal{J}$-separated, and the linear Poincaré flow on any compact invariant subset Γ of $\Lambda_X^*(U) := \Lambda_X(U) \setminus \operatorname{Sing}(X)$ is strictly $\mathcal{J}_0$-monotone, for some family $\mathcal{J}_0$ of quadratic forms equivalent to $\mathcal{J}$.

We show that, in this setting, the linear Poincaré flow of X on Γ has a hyperbolic splitting. After that, as a consequence, we show that either there are no singularities in Λ and then Λ is a hyperbolic attracting set; or, otherwise, Λ is a sectional hyperbolic attracting set, as long as the singularities are sectional hyperbolic with index compatible with the index of the attracting set.

4.1. **Strict $\mathcal{J}$-monotonicity for the linear Poincaré flow and hyperbolicity.** Strict $\mathcal{J}$-monotonicity is clearly stronger than strict $\mathcal{J}$-separation, so that on a compact invariant subset Γ of $\Lambda_X^*(U)$ the linear Poincaré flow P^t admits a dominated splitting $N^s \oplus N^u$ of N over Γ. But with strict $\mathcal{J}$-monotonicity we can say more.

Consider $X \in \mathfrak{X}^1(M)$ with a trapping region U, $\Lambda_X(U)$ its attracting set and a smooth family $\mathcal{J}$ of non-degenerate quadratic forms on U.

Proposition 4.1. *If X_t is non-negative strictly $\mathcal{J}$-separated on $\Lambda_X(U)$ and the associated linear Poincaré flow P^t over any compact invariant subset Γ of $\Lambda_X(U)^*$ is strictly $\mathcal{J}_0$-monotone for some family $\mathcal{J}_0$ of quadratic forms on $T_\Gamma M$ equivalent to $\mathcal{J}$, then Γ is a hyperbolic set for P^t.*

Proof. The property $\partial_t \mathcal{J}_0(P^t v) \mid_{t=0} > 0$ for all $v \in N_x$, $x \in \Gamma$ is equivalent to say that the quadratic form $\partial_t \mathcal{J}_{0,x}|_{N_x}$ is positive definite for all $x \in \Gamma$. This implies the existence of a function $\alpha_1 : U \to (0, +\infty)$ such that

$$\partial_t \mathcal{J}_0(P^t v) \mid_{t=0} > \alpha_1(x) \cdot \|v\|^2 > 0, \quad x \in \Gamma, v \in N_x, v \neq \vec{0}.$$

Since Γ is compact, the smoothness of $\mathcal{J}_0$ ensures the existence of $\alpha_2 > 0$ such that $|\mathcal{J}_0(v)| \leq \alpha_2 \|v\|^2$ for all $v \in N_x$, $x \in \Gamma$. Hence we obtain

$$\partial_t \mathcal{J}_0(P^t v) \mid_{t=0} \geq \alpha_1(x) \cdot \|v\|^2 \geq \frac{\alpha_1(x)}{\alpha_2} |\mathcal{J}_0(v)|$$

where $\alpha_1(x) > 0$ for all $x \in \Gamma$. Therefore we have

$$\log \frac{\mathcal{J}_0(P^t v)}{\mathcal{J}_0(v)} \geq \int_0^t \frac{\alpha_1(X^s(x))}{\alpha_2}\, ds =: H(x,t) \quad \text{for } \mathcal{J}_0\text{-positive vectors } v; \quad \text{and}$$
$$\log \frac{\mathcal{J}_0(P^t v)}{\mathcal{J}_0(v)} \leq -\int_0^t \frac{\alpha_1(X^s(x))}{\alpha_2}\, ds = -H(x,t) \quad \text{for } \mathcal{J}_0\text{-negative vectors } v.$$

From Lemma 2.24 we can compare $|\mathcal{J}_0|$ with the square of the Riemannian norm, so all that is left to do is to prove that $H(x,t)$ is unbounded for $t > 0$ and each $x \in \Gamma$.

Lemma 4.2. *For every point x in a compact invariant subset $\Gamma \subset \Lambda_X(U)^*$, we have $\lim_{t \to +\infty} H(x,t) = +\infty$.*

Proof of Lemma 4.2. If there are no singularities in Λ, then $\Lambda_X(U)^* = \Lambda_X(U)$ is compact and so there exists $\alpha_0 > 0$ such that $\alpha_1(x) \geq \alpha_0$ for all $x \in \Lambda$. This clearly implies the statement of the lemma in this case.

Let $S = \Lambda \cap S(X)$ be the set of finitely many singularities of X in Λ; we recall that we are assuming that S is formed by hyperbolic fixed points of X_t. We fix $\varepsilon > 0$ such that $\{B(\sigma, \varepsilon)\}_{\sigma \in S}$ is a pairwise disjoint family and $\Lambda \setminus B(S, \varepsilon) \neq \emptyset$, where $B(S, \varepsilon) = \cup_{\sigma \in S} B(\sigma, \varepsilon)$.

We have that $K := \Lambda \setminus B(S, \varepsilon)$ is compact and so there exists $\alpha_0 > 0$ such that $\alpha_1(x) \geq \alpha_0$ for all $x \in K$. Moreover, since the norm of the vector field X is bounded from above in Λ, there exists a minimum time $T > 0$ between consecutive visits of any orbit of $x \in \Lambda \setminus W^s(S)$ to $B(S, \varepsilon)$. That is, for $x \in \Lambda \setminus W^s(S)$, if we define sequences $t_1 < s_1 < t_2 < s_2 < \dots$ such that $X_{(t_i, s_i)}(x) \subset B(S, \varepsilon)$ and $X_{[s_i, t_{i+1}]}(x) \subset K$, then $t_{i+1} - s_i > T$, $i \geq 1$.

Since $x \in \Gamma$ and $\Gamma \cap S = \emptyset$, we have $\omega_X(x) \cap B(S, \varepsilon) = \emptyset$ for some small $\varepsilon > 0$ dependent on Γ only, in which case the sequence above terminates at some s_l for $l \geq 1$.

Hence we can write, for all $t \geq s_l$ and $t < t_{l+1}$ (if t_{l+1} does not exist, the last conditions is vacuous)

$$H(x,t) \geq \alpha_0 t_1 + \alpha_0 \sum_{i=1}^{l-1} (t_{i+1} - s_i) + (t - s_l)\alpha_0 \geq \alpha_0[(l-1)T + t_1 + (t - s_l)].$$

Either the sequences are infinite, or t_{l+1} does not exist. Hence $H(x,t)$ grows without bound when $t \to +\infty$. $\square$

Restricting x to a compact invariant subset Γ of $\Lambda_X(U)^*$, we obtain

$$\lim_{t\to-\infty} \|P^t|_{N_x^s}\| = +\infty \quad \text{and} \quad \lim_{t\to+\infty} \|P^t|_{N_x^u}\| = +\infty, \quad x \in \Gamma.$$

By well-known results, this ensures that P^t is hyperbolic over Γ; see e.g. [22]. This concludes the proof. $\square$

4.2. **Sectional hyperbolicity from the linear Poincaré flow.** Here we prove Theorem D through the following results.

Let $\Lambda = \Lambda_X(U)$ be an attracting set contained in the non-wandering set $\Omega(X)$ for a C^1 flow given by a vector field X. We recall that $\Lambda_X^*(U) = \Lambda_X(U) \setminus \mathrm{Sing}(X)$.

Theorem 4.3. *The set Λ is sectional-hyperbolic for X if, and only if, there is a neighborhood $\mathfrak{U}$ of X in $\mathfrak{X}^1(M)$ such that any compact subset Γ of $\Lambda_Y^*(U)$ is hyperbolic of index* $\mathrm{ind}(\Lambda)$ *for the linear Poincaré flow associated to any $Y \in \mathfrak{U}$ and each singularity σ of Y in the trapping region U is sectionally hyperbolic with index* $\mathrm{ind}(\sigma) \geq \mathrm{ind}(\Lambda)$.

Proof. If $E^s \oplus E^c$ is a partially hyperbolic splitting of TM over Λ, then the projections $N^s := \Pi \cdot E^s$ and $N^u := \Pi \cdot E^c$ are P^t-invariant, and N^s is uniformly contracted by P^t. Indeed, since the orthogonal projection does not increase norms, for $v \in E_x^s$ we get $\|P^t v\| = \|\Pi_{X_t(x)} DX_t(x) \cdot v\| \leq \|DX_t(x) \cdot v\|$, which is uniformly contracted for $t > 0$, as long as $X(x) \neq \vec{0}$.

Moreover, the above property persists for all vector fields Y in a small enough C^1 neighborhood of X, by the normal hyperbolic theory; see [14].

Now we assume, additionally, that E^c is sectionally expanding on Λ for X. This ensures that the continuation $E^{s,Y} \oplus E^{c,Y}$ of the partially hyperbolic splitting for C^1 close vector fields is also sectional hyperbolic. For otherwise, according to Remark 1.1, we would obtain a sequence Y_n of vector fields converging to X in the C^1 topology, a sequence x_n of points in $\Lambda_{Y^n}(U)^*$ and a sequence F_{x_n} of 2-subspaces of $E_{x_n}^{c,Y}$ such that $|\det(DY_T^n|_{F_{x_n}})| \leq 2$, $n \geq 1$ and some fixed $T > 0$. Then for a limit point x of $(x_n)_n$ in $\Lambda_X(U)$ and a limit 2-subspace F_x of the sequence F_{x_n} inside $E_x^{c,Y}$ (using the compactness of the Grassmannian over the compact set $\overline{U}$), we get $|\det(DX_T|_{F_x})| \leq 2$, contradicting the assumption of sectional-expansion on Λ.

Hence we may argue with any fixed Y close enough to X in the C^1 topology. Let us take Γ a compact subset of Λ_Y^*. For $x \in \Gamma$ the uniform expansion along N^u is obtained as follows.

Let v be a unit vector on N_x^u and let F_x be the subspace spanned by v and $X(x)$. For some $K > 0$ we have $K^{-1} \leq \|Y(x)\| \leq K$ for all $x \in \Gamma$ by compactness. Let us fix $t > 0$ and consider the basis $\{\frac{T(x)}{\|T(x)\|}, v\}$ of F_x. We note that $DY_t(F_x)$ is a bidimensional subspace F_x^t of $E^c_{Y_t(x)}$, where we take the basis $\{\frac{Y(Y_t(x))}{\|Y(Y_t(x)x)\|}, w_t\}$, with

$$w_t := \frac{\Pi_{Y_t(x)} \cdot DY_t(x)(v)}{\|\Pi_{Y_t(x)} \cdot DY^t(x)(v)\|} \quad \text{belonging to} \quad N^u_{Y_t(x)}.$$

With respect to these orthonormal bases we have

$$DY_t|_{F_x} = \begin{bmatrix} \frac{\|Y(Y_t(x))\|}{\|Y(x)\|} & \star \\ 0 & \Delta \end{bmatrix},$$

because the flow direction is invariant. Hence

$$\det\left(DY_t|_{E_x^c}\right) = \frac{\|Y(Y_t(x))\|}{\|Y(x)\|}\Delta \leq K^2\Delta$$

for some $K > 0$ depending only on $\Gamma \subset \Lambda_Y(U)^*$, and

$$\begin{aligned} \|P_x^{Y,t} \cdot v\| &= \|\Pi_{X_t(x)} \cdot DY_t(x)(v)\| = \|\Delta \cdot w\| = |\Delta| \\ &\geq K^{-2}|\det(DY_t|_{F_x})| \geq K^{-2}Ce^{\lambda t}. \end{aligned}$$

This proves that N^u is uniformly expanded by the linear Poincaré flow $P^{Y,t}$ over Γ. Moreover, for every singularity $\sigma \in \Lambda$ we have $T_\sigma M = E^s_\sigma \oplus E^u_\sigma$ a sectional hyperbolic splitting, thus $\operatorname{ind}(\sigma) \geq \operatorname{ind}(\Lambda)$; in fact, sectional expansion on E^c_σ ensures that either $\operatorname{ind}(\sigma) = \operatorname{ind}(\Lambda)$ or $\operatorname{ind}(\sigma) = \operatorname{ind}(\Lambda) + 1$.

Reciprocally, let us assume that Λ is a compact attracting set with isolating neighborhood U such that: the linear Poincaré flow over any compact subset $\Gamma \subset \Lambda_Y^*(U)$ is hyperbolic with constant index $\operatorname{ind}(\Lambda)$, for all Y in a C^1 neighborhood $\mathcal{U}$ of X; and that the singularities σ in U for each $Y \in \mathcal{U}$ are sectionally hyperbolic with index $\operatorname{ind}(\sigma) \geq \operatorname{ind}(\Lambda)$. In particular, the index of all periodic orbits of U for $Y \in \mathcal{U}$ is constant $\operatorname{ind}(\Lambda)$, and the flows in $\mathcal{U}$ are homogeneous. Hence, every periodic orbit in U for Y is hyperbolic with uniform bounds on the expansion and contraction on the period and, moreover, admits a sectional-hyperbolic splitting of the tangent bundle with constant index $\operatorname{ind}(\Lambda)$ and with angle between the stable and central directions uniformly bounded away from zero; see [4, Section 5.4.1].

This is enough to deduce that the tangent bundle on $\Lambda_Y(U) \cap \Omega(X)$ admits a partially hyperbolic splitting $E^s \oplus E^c$ with index $\dim E^s = \operatorname{ind}(\Lambda)$, since the assumption on the index of the singularities ensures that the partial hyperbolic splitting on every periodic orbit for each flow $Y \in \mathcal{U}$ can be extended to a partially hyperbolic splitting on the entire non-wandering set intersected with Λ, including the singularities; see [4, Sections 5.4.2].

Having this property robustly on $\mathcal{U}$ with sectional hyperbolicity on periodic orbits implies that the subbundle E^c is sectionally expanding, for $Y \in \mathcal{U}$; see [4, Sections 5.4.3].

But since we assume that $\Lambda_X(U) \subset \Omega(X)$, the proof is complete. □

Now Proposition 4.1 shows that strict $\mathcal{J}$-monotonicity for the linear Poincaré flow over a compact invariant subset implies hyperbolicity. Together with Theorem 4.3 we conclude the proof of sufficiency in Theorem D, for the non-wandering part $\Lambda_X(U) \cap \Omega(X)$ of Λ.

Moreover, using ideas similar to the ones in the proof of Lemma 3.11 we conclude that we can extend the above conclusion for the entire maximal invariant set $\Lambda_X(U)$.

This completes the proof that strict J-monotonicity of the linear Poincaré flow implies sectional hyperbolicity, which is half of the statement of Theorem D.

4.3. **Sectional hyperbolicity implies strict $\mathcal{J}$-monotonicity on compacts for the linear Poincaré flow.** Now we prove that having a sectional hyperbolic splitting implies that there exists a field $\mathcal{J}$ of non-degenerate and indefinite quadratic forms with constant index equal to the dimension of the contracting direction, such that the linear Poincaré flow is strictly $\mathcal{J}_0$-monotonous on every compact invariant set without singularities, for some equivalent family $\mathcal{J}_0$ of forms, completing the proof of Theorem D.

Let $\Lambda = \Lambda_X(U)$ be a compact maximal invariant set admitting a sectional hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$. As noted in the first part of the proof of Theorem 4.3, the existence of sectional hyperbolic splitting is a robust property: there exists a neighborhood $\mathcal{U}$ of X in the C^1 topology in $\mathfrak{X}^1(M)$ such that all $Y \in \mathcal{U}$ have a maximal invariant subset $\Lambda_Y(U)$ which is also sectional hyperbolic. Hence the results we obtain below hold robustly in a neighborhood of X.

We have already shown, in Section 2.5, how to construct a field $\mathcal{J}$ of quadratic forms such that X is strictly $\mathcal{J}$-separated on a neighborhood $V \subset U$ of Λ satisfying for some $\lambda > 0$ and all $x \in \Lambda$ and $t > 0$

$$|DX_t v^+| = \sqrt{\mathcal{J}(DX_t v^+)} \ge e^{\lambda t}\sqrt{\mathcal{J}(DX_t v^-)} = e^{\lambda t}|DX_t v^-|, \quad v^- \in E^s_x, v^+ \in E^c_x, |v^\pm| = 1.$$

The results in [13] extend the properties of adapted metrics to partial hyperbolic splittings, in such a way that we can also obtain for all $t > 0$

$$|DX_t v^-| \le e^{-\lambda t}, \quad v^- \in E^s_x, |v^-| = 1.$$

On $\Lambda^* = \Lambda \setminus \mathrm{Sing}(X)$ we define $N^s = \prod \cdot E^s$ and $N^u = \prod \cdot E^c$, where $\prod$ is the projection of the tangent bundle onto the pseudo-orthogonal complement N of X with respect to $\mathcal{J}$. We note that since $P^t = \prod \cdot DX_t$

$$|P^t\,|_{N^s_x}| \cdot |P^{-t}\,|_{N^u_{X_t(x)}}| \le |DX_t\,|_{E^s_x}| \cdot |DX_{-t}\,|_{E^c_{X_t(x)}}| \le e^{-\lambda t}, \quad \text{and} \tag{4.1}$$

$$|P^t\,|_{N^s_x}| \le |DX_t\,|_{E^s_x}| \le e^{-\lambda t}, \quad x \in \Lambda^*, t > 0 \tag{4.2}$$

so that the linear Poincaré flow has a partially hyperbolic splitting over Λ^*.

The assumption of sectional expansion ensures that, if we fix any unit vector $v \in N^u_x$ for $x \in \Lambda^*$, then for some $C, \lambda > 0$ and every $t > 0$

$$
\begin{aligned}
Ce^{\lambda t} \le |\det DX_t \mid_{\mathrm{span}\{X(x),v\}}| &= \frac{\mathrm{vol}(DX_t v, X(X_t(x)))}{\mathrm{vol}(X(x), v)} \\
&= \frac{|X(X_t(x))|}{|X(x)|}|DX_t v| \sin\sphericalangle(DX_t v, X(X_t(x))) = \frac{|X(X_t(x))|}{|X(x)|}|P^t v|.
\end{aligned}
$$

Since we are in a compact set we have $0 < c_0 = \sup_{z\in\Lambda}|X(z)| < \infty$ and so

$$
|P^t v| \ge \frac{C|X(x)|}{c_0} e^{\lambda t}, \quad x \in \Lambda^*, v \in N^u_x, |v| = 1, t > 0. \tag{4.3}
$$

We write $c(x) := C|X(x)|/c_0$ and note that $0 < c(x) \le 1$ by letting $t \to 0$ in the above inequality.

We restrict now to the case of a compact invariant subset Γ of Λ^*. In this case $c(x) \ge c_1 > 0$ for all $x \in \Gamma$ and $N^s_\Gamma \oplus N^u_\Gamma$ is a uniformly hyperbolic splitting for P^t. We can then obtain an adapted Riemannian metric for this splitting following [13] and define a field $\mathcal{J}_0$ of quadratic forms using this adapted metric as in Section 2.5.1. With respect to the adapted metric we obtain both (4.1) and (4.2), and also (4.3) but with unit constants multiplying the exponential. From Remark 2.18, since $\mathcal{J}$ and $\mathcal{J}_0$ have the same signs on the E^s_Γ and E^c_Γ, then $\mathcal{J}_0 \sim \mathcal{J}$.

We show that P^t is strictly $\mathcal{J}_0$-monotonous. We consider a vector $v \in N_x$ with $v = v^- + v^+, v^- \in N^s_x, v^+ \in N^u_x$ and $\mathcal{J}_0(v^+) - \mathcal{J}_0(v^-) = |v^+|^2 + |v^-|^2 = 1$ for $x \in \Gamma$, and the norm of its image under the linear Poincaré flow. Since $P^t v = P^t v^+ + P^t v^-$ and N^s and N^u are P^t-invariant, if $v^+ \ne \vec{0}$ we get for $t > 0$

$$
\begin{aligned}
\mathcal{J}_0(P^t v) = \mathcal{J}_0(P^t v^+) + \mathcal{J}_0(P^t v^-) &= \mathcal{J}_0(P^t v^+) \cdot \left(1 + \frac{\mathcal{J}_0(P^t v^-)}{\mathcal{J}_0(P^t v^+)}\right) \\
&\ge e^{2\lambda t}\mathcal{J}_0(v^+)\left(1 + e^{-2\lambda t}\frac{\mathcal{J}_0(v^-)}{\mathcal{J}_0(v^+)}\right),
\end{aligned}
$$

since $\mathcal{J}(P^t v^-) < 0$. We note that the value of the left hand side and the right hand side above are the same at $t = 0$. Moreover the derivative of the right hand side with respect to t at $t = 0$ equals

$$
\left[2\lambda e^{2\lambda t}\mathcal{J}_0(v^+)\left(1 + e^{-2\lambda t}\frac{\mathcal{J}_0(v^-)}{\mathcal{J}_0(v^+)}\right) - e^{2\lambda t}\mathcal{J}_0(v^+) \cdot 2\lambda e^{-2\lambda t}\frac{\mathcal{J}_0(v^-)}{\mathcal{J}_0(v^+)}\right]\Bigg|_{t=0} = 2\lambda\mathcal{J}_0(v^+) > 0.
$$

Hence we conclude that $\partial_t \mathcal{J}_0(P^t v) \mid_{t=0} \ge 2\lambda\mathcal{J}_0(v^+) > 0$ when v has a non-zero positive component. In the remaining case, $v = v^-$ we obtain (again because $\mathcal{J}_0(P^t v^-) < 0$)

$$
\mathcal{J}_0(P^t v^-) \ge e^{-2\lambda t}\mathcal{J}_0(v^-)
$$

with the same value at $t = 0$ on both sides, so that $\partial_t \mathcal{J}_0(P^t v^-) \mid_{t=0} \ge -2\lambda\mathcal{J}_0(v^-) > 0$ also in this case.

We have proved that for all vector fields Y sufficiently C^1 close to X and for every compact invariant subset Γ of $\Lambda_Y(U)^*$, we can find a field $\mathcal{J}_0$ of quadratic forms equivalent to $\mathcal{J}$ over Γ such that P^t_Y is strictly $\mathcal{J}_0$-monotonous, as claimed in Theorem D.

This together with Theorem 4.3 completes the proof of Theorem D.

4.4. **Criteria for $\mathcal{J}$-monotonicity of the linear Poincaré flow.** Here we prove Proposition 1.4. We have already characterized partial hyperbolicity using the notion of $\mathcal{J}$-separation, or the existence of infinitesimal Lyapunov functions, which depend only on the vector field X and its derivative DX. To present a characterization of sectional hyperbolicity along the same lines, we must use the conclusion of Theorem D, and obtain a criterion for the linear Poincaré flow to be $\mathcal{J}$-monotonous.

The condition of $\mathcal{J}$-monotonicity for the linear Poincaré flow can be expressed using only the vector field X and its space derivative DX.

Recall that a self-adjoint operator is said to be (positive) non-negative if all eigenvalues are (positive) non-negative.

Lemma 4.4. *Let X be a $\mathcal{J}$-non-negative (positive) vector field on U. Then, the Linear Poincaré Flow is (strictly) $\mathcal{J}$-monotone if, and only if, the operator*

$$\hat{J}_x := DX(x)^* \cdot \Pi_x^* J \Pi_x + \Pi_x^* J \Pi_x \cdot DX(x)$$

is a non-negative (positive) self-adjoint operator.

Here, we consider Π^* as the adjoint operator of the orthogonal projection Π in the definition of the linear Poincaré flow.

The conditions above are again consequence of the corresponding results for linear multiplicative cocycles over flows, as explained in Section 4.

Proof. We shall prove only the positive case, once the non-negative is similar.

We denote by $|v| := \langle Jv, v\rangle^{1/2}$ the $\mathcal{J}$-norm of a vector v and observe that we can write

$$\Pi_{X_t(x)} v := v - \langle Jv, \frac{X(X_t(x))}{|X(X_t(x))|}\rangle \frac{X(X_t(x))}{|X(X_t(x))|},$$

for all $v \in T_xM$ with $X(X_t(x)) \neq \vec{0}$ and $t \geq 0$. Then, to conclude that $\mathcal{J}(P^t v) > \mathcal{J}(v)$ for every $v \in N_x$, $x \in U, X(x) \neq \vec{0}$ it is enough to prove

$$\partial_t \mathcal{J}(P^t v) > 0 \quad \text{for every} \quad v \in T_xM,\ X(X_t(x)) \neq \vec{0} \quad \text{and} \quad t \geq 0. \tag{4.4}$$

Reciprocally, if we have that $\mathcal{J}(P^t v) > \mathcal{J}(v)$ for every $v \in N_x$, $x \in U, X(x) \neq \vec{0}$, then we also must have (4.4).

Now the above derivative can be written, just like in the previous sections

$$\langle J \cdot \Pi_{X_t(x)} DX_t v, \partial_t(\Pi_{X_t(x)} DX_t v)\rangle + \langle J \cdot \partial_t(\Pi_{X_t(x)} DX_t v), \Pi_{X_t(x)} DX_t v\rangle. \tag{4.5}$$

To expand the above expression, we note that

$$\partial_t(\Pi_{X_t(x)} DX_t v) = \partial_t \left(DX_t v - \langle J \cdot DX_t v, \frac{X(X_t(x))}{|X(X_t(x))|}\rangle \frac{X(X_t(x))}{|X(X_t(x))|} \right)$$

can be written in the following way,

$$\begin{aligned}&DX(X_t(x))DX_tv+\langle J\cdot DX_tv,\frac{X(X_t(x))}{|X(X_t(x))|}\rangle\cdot\partial_t\frac{X(X_t(x))}{|X(X_t(x))|}\\&\qquad-\Big(\langle J\cdot DX(X_t(x))DX_tv,\frac{X(X_t(x))}{|X(X_t(x))|}\rangle+\langle J\cdot DX_tv,\partial_t\frac{X(X_t(x))}{|X(X_t(x))|}\rangle\Big)\frac{X(X_t(x))}{|X(X_t(x))|}.\end{aligned}$$

Since $\partial_t\frac{X(X_t(x))}{|X(X_t(x))|}$ equals

$$-\langle\frac{X(X_t(x))}{|X(X_t(x))|},DX(X_t(x))\frac{X(X_t(x))}{|X(X_t(x))|}\rangle\cdot\frac{X(X_t(x))}{|X(X_t(x))|}+DX(X_t(x))\frac{X(X_t(x))}{|X(X_t(x))|}$$

and must be $\mathcal{J}$-orthogonal to the flow direction at $X_t(x)$, then this last expression is the projection on $N_{X_t(x)}$ as follows

$$\partial_t\frac{X(X_t(x))}{|X(X_t(x))|}=\big(\Pi_{X_t(x)}DX(X_t(x))\big)\cdot\frac{X(X_t(x))}{|X(X_t(x))|}$$

Now replacing $X_t(x)$ by z throughout and the vector $X(z)$ $\mathcal{J}$-normalized by $\hat{X}(z)$ we obtain the following expression for the derivative of P^tv

$$\begin{aligned}&DX(z)DX_tv-\langle J\cdot DX_tv,\hat{X}(z)\rangle\cdot\Pi_zDX(z)\hat{X}(z)\\&\qquad-\Big(\langle J\cdot DX(z)DX_tv,\hat{X}(z)\rangle+\langle J\cdot DX_tv,\Pi_zDX(z)\hat{X}(z)\rangle\Big)\hat{X}(z)\end{aligned}$$

or, easier for a geometrical interpretation

$$\begin{aligned}&DX(z)DX_tv-\langle J\cdot DX(z)DX_tv,\hat{X}(z)\rangle\hat{X}(z)\\&\qquad-\langle J\cdot DX_tv,\hat{X}(z)\rangle\cdot\Pi_zDX(z)\hat{X}(z)-\langle J\cdot DX_tv,\Pi_zDX(z)\hat{X}(z)\rangle\hat{X}(z).\end{aligned}$$

We observe that the first line above is the projection on N_z of $DX(z)DX_tv$ so we have that ∂_tP^tv equals

$$\Pi_zDX(z)DX_tv-\langle J\cdot DX_tv,\hat{X}(z)\rangle\Pi_zDX(z)\hat{X}(z)-\langle J\cdot DX_tv,\Pi_zDX(z)\hat{X}(z)\rangle\hat{X}(z).$$

In the expression (4.5), we take the $\mathcal{J}$-(inner)-product with a vector on N_z, so the $\hat{X}$ component above contributes nothing to the final result. Therefore (4.5) becomes

$$\begin{aligned}&\langle J\cdot\Pi_zDX_tv,\Pi_zDX(z)DX_tv\rangle+\langle J\cdot\Pi_zDX(z)DX_tv,\Pi_zDX_tv\rangle\\&\qquad-\langle J\cdot DX_tv,\hat{X}(z)\rangle\big(\langle J\cdot\Pi_zDX_tv,\Pi_zDX(z)\hat{X}(z)\rangle+\langle J\cdot\Pi_zDX(z)\hat{X}(z),\Pi_zDX_tv\rangle\big)\end{aligned}$$

and using the adjoint of $DX(z)=DX(X_t(x))$ we define $\hat{J}=\hat{J}_x:=\big(\Pi_xDX(x)\big)^*J\Pi_x+\Pi_x^*J\Pi_xDX(x)$ and obtain

$$\partial_tP^tv=\langle\hat{J}_{X_t(x)}DX_tv,DX_tv\rangle-\langle J\cdot DX_tv,\hat{X}(X_t(x))\rangle\cdot\langle\hat{J}\cdot DX_tv,\hat{X}(X_t(x))\rangle.$$

Letting $t=0$, since $\langle J\cdot v,\hat{X}(x)\rangle=0$ for $v\in N_x$, we arrive at

$$\partial_t\big(P^tv\big)\mid_{t=0}=\langle\hat{J}_xv,v\rangle-\langle J\cdot v,\hat{X}(x)\rangle\cdot\langle\hat{J}\cdot v,\hat{X}(x)\rangle=\langle\hat{J}_xv,v\rangle.\tag{4.6}$$

We conclude that condition (4.4) is equivalent to

$$\langle \hat{J}_{X_t(x)} v, v\rangle > 0, \quad v \in N_{X_t(x)}, \tag{4.7}$$

that is, $\hat{J}_x$ *is a positive definite self-adjoint operator on* N_x *for each* $x \in U$ *with* $X(x) \neq \vec{0}$. Indeed, by the flow property of P^t we have, for all $s > 0$

$$\partial_t \mathfrak{J}(P^{t+s} v) \mid_{t=0} = \partial_t \mathfrak{J}\big(P^t(P^s v)\big) \mid_{t=0} > 0 \quad \text{because} \quad P^s v \in N_{X^s(x)}$$

and $P^s : N_x \to N_{X^s(x)}$ is an isomorphism. $\square$

So (4.7) is the condition that the vector field and its derivative must satisfy in U, except at singularities, so that the Linear Poincaré Flow admits a hyperbolic splitting.

This concludes the proof of Proposition 1.4.

References

[1] V. M. Alekseev. Quasirandom dynamical systems. I. Quasirandom diffeomorphisms. *Mat. Sb. (N.S.)*, 76 (118):72–134, 1968.

[2] V. M. Alekseev. Quasirandom dynamical systems. II. One-dimensional nonlinear vibrations in a periodically perturbed field. *Mat. Sb. (N.S.)*, 77 (119):545–601, 1968.

[3] V. M. Alekseev. Quasirandom dynamical systems. III. Quasirandom vibrations of one-dimensional oscillators. *Mat. Sb. (N.S.)*, 78 (120):3–50, 1969.

[4] V. Araújo and M. J. Pacifico. *Three-dimensional flows*, volume 53 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2010. With a foreword by Marcelo Viana.

[5] M. Benedicks and R. Moeckel. An attractor for certain Hénon maps. Preprint ETH.

[6] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III.

[7] C. Bonatti, N. Gourmelon, and T. Vivier. Perturbations of the derivative along periodic orbits. *Ergodic Theory Dynam. Systems*, 26(5):1307–1337, 2006.

[8] C. Bonatti, A. Pumariño, and M. Viana. Lorenz attractors with arbitrary expanding dimension. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(8):883–888, 1997.

[9] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[10] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[11] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29:181–202, 1975.

[12] C. I. Doering. Persistently transitive vector fields on three-dimensional manifolds. In *Procs. on Dynamical Systems and Bifurcation Theory*, volume 160, pages 59–89. Pitman, 1987.

[13] N. Gourmelon. Adapted metrics for dominated splittings. *Ergodic Theory Dynam. Systems*, 27(6):1839–1849, 2007.

[14] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*, volume 583 of *Lect. Notes in Math.* Springer Verlag, New York, 1977.

[15] T. J. Hunt and R. S. MacKay. Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor. *Nonlinearity*, 16(4):1499, 2003.

[16] A. Katok. Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems. *Ergodic Theory Dynam. Systems*, 14(4):757–785, 1994. With the collaboration of Keith Burns.

[17] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopeadia Appl. Math.* Cambridge University Press, Cambridge, 1995.
[18] J. Lewowicz. Lyapunov functions and topological stability. *J. Differential Equations*, 38(2):192–209, 1980.
[19] J. Lewowicz. Expansive homeomorphisms of surfaces. *Bol. Soc. Brasil. Mat. (N.S.)*, 20(1):113–133, 1989.
[20] E. N. Lorenz. Deterministic nonperiodic flow. *J. Atmosph. Sci.*, 20:130–141, 1963.
[21] A. I. Mal′cev. *Foundations of linear algebra*. Translated from the Russian by Thomas Craig Brown; edited by J. B. Roberts. W. H. Freeman & Co., San Francisco, Calif.-London, 1963.
[22] R. Mañé. An ergodic closing lemma. *Annals of Math.*, 116:503–540, 1982.
[23] R. Markarian. Billiards with Pesin region of measure one. *Comm. Math. Phys.*, 118(1):87–97, 1988.
[24] R. Metzger and C. Morales. Sectional-hyperbolic systems. *Ergodic Theory and Dynamical System*, 28:1587–1597, 2008.
[25] C. A. Morales. Examples of singular-hyperbolic attracting sets. *Dynamical Systems, An International Journal*, 22(3):339–349, 2007.
[26] C. A. Morales, M. J. Pacifico, and E. R. Pujals. Singular hyperbolic systems. *Proc. Amer. Math. Soc.*, 127(11):3393–3401, 1999.
[27] C. A. Morales, M. J. Pacifico, and E. R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Ann. of Math. (2)*, 160(2):375–432, 2004.
[28] S. Newhouse. Cone-fields, domination, and hyperbolicity. In *Modern dynamical systems and applications*, pages 419–432. Cambridge Univ. Press, Cambridge, 2004.
[29] J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems*. Springer Verlag, 1982.
[30] V. P. Potapov. The multiplicative structure of J-contractive matrix functions. *Amer. Math. Soc. Transl. (2)*, 15:131–243, 1960. Translation of Trudy Moskovskogo Matematičeskogo Obščestva 4 (1955), 125–236.
[31] V. P. Potapov. Linear-fractional transformations of matrices. In *Studies in the theory of operators and their applications (Russian)*, pages 75–97, 177. "Naukova Dumka", Kiev, 1979.
[32] C. Robinson. *Dynamical systems*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, Second edition, 1999. Stability, symbolic dynamics, and chaos.
[33] C. Robinson. *An introduction to dynamical systems: continuous and discrete*. Pearson Prentice Hall, Upper Saddle River, NJ, 2004.
[34] L. Salgado. *Sobre hiperbolicidade fraca para fluxos singulares*. PhD thesis, UFRJ, Rio de Janeiro, 2012.
[35] M. Shub. *Global stability of dynamical systems*. Springer Verlag, 1987.
[36] Y. Sinai. Markov partitions and C-diffeomorphisms. *Func. Anal. and Appl.*, 2:64–89, 1968.
[37] S. Smale. Differentiable dynamical systems. *Bull. Am. Math. Soc.*, 73:747–817, 1967.
[38] S. Strogatz. *Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering*. Studies in nonlinearity. Perseus Books, Reading, Massachusetts, 1st edition, 1994.
[39] W. Tucker. The Lorenz attractor exists. *C. R. Acad. Sci. Paris*, 328, Série I:1197–1202, 1999.
[40] W. Tucker. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.*, 2(1):53–117, 2002.
[41] M. Viana. What's new on Lorenz strange attractor. *Mathematical Intelligencer*, 22(3):6–19, 2000.
[42] T. Vivier. Flots robustement transitifs sur les variétés compactes. *C. R. Math. Acad. Sci. Paris*, 337(12):791–796, 2003.
[43] M. Wojtkowski. Invariant families of cones and Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 5(1):145–161, 1985.
[44] M. P. Wojtkowski. Magnetic flows and Gaussian thermostats on manifolds of negative curvature. *Fund. Math.*, 163(2):177–191, 2000.
[45] M. P. Wojtkowski. W-flows on Weyl manifolds and Gaussian thermostats. *J. Math. Pures Appl. (9)*, 79(10):953–974, 2000.

[46] M. P. Wojtkowski. Monotonicity, J-algebra of Potapov and Lyapunov exponents. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 499–521. Amer. Math. Soc., Providence, RI, 2001.

[47] M. P. Wojtkowski. A simple proof of polar decomposition in pseudo-Euclidean geometry. *Fund. Math.*, 206:299–306, 2009.

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